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Statistical Estimates for the Navier–Stokes Equations and the Kraichnan Theory of 2-D Fully Developed Turbulence

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A mathematical formulation of the Kraichnan theory for 2-D fully developed turbulence is given in terms of ensemble averages of solutions to the Navier–Stokes equations. A simple condition is given for the enstrophy cascade to hold for wavenumbers just beyond the highest wavenumber of the force up to a fixed fraction of the dissipation wavenumber, up to a logarithmic correction. This is followed by partial rigorous support for Kraichnan's eddy breakup mechanism. A rigorous estimate for the total energy is found to be consistent with Kraichnan's theory. Finally, it is shown that under our conditions for fully developed turbulence the fractal dimension of the attractor obeys a sharper upper bound than in the general case.

KEY WORDS: Navier–Stokes; turbulence.

INTRODUCTION

Much of the success of conventional theoretical physics (electromagnetic theory, quantum mechanics, acoustics, etc.) is owed to the firm mathematical infrastructure underlying that field, namely the theory of linear partial differential equations, and more generally the theory of linear operators. In contrast until recently such a common phenomenon as the high rate of flow of water and the accompanying turbulence had no comparable firm foundation. Albeit the underlying equations modeling this phenomenon, viz., the Navier–Stokes equations has been well known as a model, its

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nonlinear nature, together with the observed random behavior of the flow precluded obtaining explicit solutions of those equations. Thus the theory of turbulence had to rely to a large extent on phenomenological, intuitive arguments and, more recently on limited numerical solutions of the underlying equations and their approximations.

While the results of the intuitive arguments can be compared to some extent with the experimental studies of flows in 3-D, the paucity of data for turbulent flows in 2-D leaves the theoretical considerations of such flows open to some questions. For instance, the attractive, physically reasonable theory of 2-D turbulence due to Kraichnan(1) rests on some more or less explicit assumptions, which are difficult, if not impossible, to test experimentally. In particular, there is the question of the extent to which that theory is in fact consistent with the Navier–Stokes equations in 2-D. Modern methods of functional analysis as applied to nonlinear partial differential equations, together with the theory of statistical solutions of such equations offer some powerful, even if somewhat incomplete tools for answering those questions. It is the purpose of this paper to explore the correspondence between the rigorous properties of the solutions of 2-D Navier–Stokes equations, and the largely phenomenologically motivated assumptions of Kraichnan's theory. Admittedly, because of the current limitations of the available mathematical tools, we are forced to apply some heuristic arguments, but even those are closer to mathematical rigor than the conventional approach used hitherto. As will be seen subsequently, much of the results of ref. 1 are in fact derivable from the underlying equations, but some new insights are gained in the process of this study.

In Section 1 we present some definitions, notation, and recall some relevant results from previous efforts. Section 2 contains the basic concepts needed to follow the statistical properties of the solutions of Navier–Stokes equations. In particular, the relationship between time averages and statistical ensemble averages is explored. It is important to emphasize here that, even though the explicit expressions for the statistics of the turbulent velocity field are not available, much can be said about the nature of the statistical solutions. A number of relations between averaged terms in the Navier–Stokes equations are also derived in Section 2.

In Section 3 we prove that the rate of enstrophy transfer from low to high modes dominates that from high to low modes. Unfortunately, the result is proved to hold only deep in the dissipation range (see Remark 5.10). As a consequence it is merely suggestive of an eddy breakup mechanism in the inertial range proposed by Kraichnan. The tenets of the Kraichnan theory for fully developed two-dimensional turbulence are first recalled in general physical terms in Section 4. For completeness we reproduce the universality argument of Kolmogorov, Batchelor and Kraichnan for the forms of both the energy per unit mass of eddies and the dissipative wavenumber in terms of the total enstrophy dissipation rate and respectively, wavenumber and viscosity. We then quantify energy, enstrophy, and enstrophy dissipation rate in terms of the solution to the Navier–Stokes equation and present Kraichnan's alternative approach to the form of the energy per unit mass, which assumes the more specific eddy breakup mechanism. Finally it is shown that for the Kraichnan theory to hold, the Grashof number must be large.

Rigorous partial support for the Kraichnan theory is presented in Section 5. This includes a range for the cascade of enstrophy, provided a certain wavenumber taken as the ratio of the averages of two norms is sufficiently large compared to the highest wavenumber in the force. This alternative wavenumber is then shown to be bounded by the Kraichnan dissipative wavenumber up to a nondimensional constant factor, depending only on the highest wavenumber of the force and the domain length. Then assuming both Kraichnan's theory and his particular form of the energy spectrum, a sharper estimate is obtained for this alternative wavenumber with the constant in the previous estimate replaced by a term which decays as the reciprocal of the log of the Grashof number. The dissipation due to viscosity from this alternative wavenumber to the dissipative wavenumber is comparable to the total dissipation. A rigorous estimate for the total energy is found to be consistent with one deduced from the Kraichnan theory.

A completely mathematical formulation of the Kraichnan theory is given in Section 6. One of the conditions is reformulated in terms of the force, which simplifies to a readily verifiable condition in the case of two forcing modes. We finish with a rigorous estimate for the Landau–Lifschitz degrees of freedom. If the Kraichnan theory is assumed to hold, this estimate results in an improvement over previous estimates (in terms of the Grashof number) for the dimension of the global attractor of the Navier–Stokes equations.

While this work supplements the results on 2-D turbulence in ref. 2, we tried to keep it as self-contained as possible. Similar results have recently been obtained for fully developed 3-D turbulence (see refs. 3 and 4). Unlike other rigorous studies of turbulence made in terms of the space variables (see, for example, ref. 5), the work here is done entirely in terms of the wavenumber, in an effort to provide rigorous support for Kraichnan's approach.

During the final polishing stage of this paper our coauthor and friend, Oscar P. Manley, passed away. We dedicate this paper to his memory, as an homage to the inspirational role he played in much of our work.

1. PRELIMINARIES

The incompressible Navier–Stokes equations

$$\frac{\partial u}{\partial t} - \nu\,\Delta u + (u \cdot \nabla)\,u + \nabla p = F,$$

$$\operatorname{div} u = 0$$

$$\int_\Omega u\,dx = 0$$

$$u(x, 0) = u_0(x)$$

with periodic boundary conditions in $\Omega = [0, L]^2$ can be written as a differential equation in a certain Hilbert space H (see refs. 6 and 7),

$$\frac{du}{dt} + \nu A u + B(u, u) = f, \qquad u \in H \tag{1.1}$$

The phase space H is the subspace of $L^2(\Omega)^2$ consisting of the closure of the set of all $\mathbb{R}^2$-valued trigonometric polynomials u such that

$$\nabla \cdot u = 0 \qquad \text{and} \qquad \int_\Omega u(x)\,dx = 0$$

The bilinear operator B is defined as

$$B(u, v) = \mathscr{P}((u \cdot \nabla)\,v)$$

where $\mathscr{P}$ is the Helmholtz–Leray projection onto H, that is, the orthogonal projection of $L^2(\Omega)^2$ onto H. The scalar product in H is taken to be

$$(u, v) = \int_\Omega u(x) \cdot v(x)\,dx, \qquad \text{where} \quad a \cdot b = a_1 b_1 + a_2 b_2$$

with associated norm

$$|u| = (u, u)^{1/2} = \left(\int_\Omega u(x) \cdot u(x)\,dx\right)^{1/2}$$

The operator $A = -\Delta$ is self-adjoint and its eigenvalues are of the form

$$\left(\frac{2\pi}{L}\right)^2 k \cdot k, \qquad \text{where} \quad k \in \mathbb{Z}^2 \backslash \{0\}$$

We denote by $0 < \lambda_0 = (\frac{2\pi}{L})^2 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ these eigenvalues arranged in an increasing order and counted according to their multiplicities, and write $w_0, w_1, w_2,\ldots,$ for the corresponding normalized eigenvectors (i.e., $|w_j| = 1$ for $j = 0, 1, 2,\ldots$).

The positive roots of A are defined by linearity from

$$A^\alpha w_j = \lambda_j^\alpha w_j, \qquad \text{for} \quad j = 0, 1, 2,\ldots$$

on the set

$$\mathscr{D}_{A^\alpha} = \left\{ u \in H : \sum_{j=0}^{\infty} \lambda_j^{2\alpha} (u, w_j)^2 < \infty \right\}$$

We will write $V = \mathscr{D}_{A^{1/2}}$ and the natural norm on V will be

$$\|u\| = |A^{1/2} u| = \left(\int_\Omega \sum_{j=1}^{2} \frac{\partial}{\partial x_j} u(x) \cdot \frac{\partial}{\partial x_j} u(x) \, dx \right)^{1/2} = \left(\sum_{j=0}^{\infty} \lambda_j (u, w_j)^2 \right)^2$$

Given the periodic boundary conditions, we may express an element in H as a Fourier series

$$u(x) = \sum_{k \in \mathbb{Z}^2} a_k e^{i\kappa_0 k \cdot x} \tag{1.2}$$

where

$$\kappa_0 = \lambda_0^{1/2} = \frac{2\pi}{L} \tag{1.3}$$

$a_0 = 0$, $a_k^* = a_{-k}$, and due to incompressibility, $k \cdot a_k = 0$. We shall associate to each term in (1.2) a *wavenumber* $\kappa_0 \, |k|$. Parseval's identity reads as

$$|u|^2 = L^2 \sum_{k \in \mathbb{Z}^2} a_k \cdot a_{-k} = L^2 \sum_{k \in \mathbb{Z}^2} |a_k|^2$$

(we have also used $|\cdot|$ for the modulus of a vector in $\mathbb{C}^2$; we assume that the meaning will be clear from the context) as well as

$$(u, v) = L^2 \sum_{k \in \mathbb{Z}^2} a_k \cdot b_{-k}$$

for $v = \sum b_k e^{i\kappa_0 k \cdot x}$. We define projectors P_κ: $H \to \operatorname{span}\{w_j \mid \lambda_j \leqslant \kappa^2\}$ by

$$P_\kappa u = \sum_{\kappa_0 \, |k| \leqslant \kappa} a_k e^{i\kappa_0 k \cdot x}$$

where u has the expansion in (1.2), along with $Q_\kappa = I - P_\kappa$. For convenience, we define components of u by a range in wavenumber

$$u_{\kappa, \kappa'} = (P_{\kappa'} - P_\kappa)\, u$$

for $0 \leqslant \kappa < \kappa'$, with the convention that $u_{\kappa, \infty} = u$ for all $0 \leqslant \kappa < \kappa_0$.

Recall the orthogonality relations of the bilinear term (see, for instance, ref. 7)

$$(B(u, v), w) = -(B(u, w), v) \tag{1.4}$$

and in two space dimensions only,

$$(B(u, u), Au) = 0 \tag{1.5}$$

Recall as well the strong form of enstrophy invariance (see, for instance, ref. 8)

$$(B(Av, v), u) = (B(u, v), Av) \tag{1.6}$$

A number of other relations we will use below follow easily from these. By orthogonality, we have

$$\begin{aligned} 0 &= (B(tu+v, tu+v), A(tu+v)) \\ &= (B(v, v), Av) + t[(B(u, v), Av) + (B(v, u), Av) + (B(v, v), Au)] \\ &\quad + \{\text{terms of degree 2 and 3 in } t\} \end{aligned}$$

which yields the relation

$$(B(u, v), Av) + (B(v, u), Av) + (B(v, v), Au) = 0 \tag{1.7}$$

Applying (1.6) and (1.4) to (1.7) we obtain

$$(B(Av, v), u) - (B(v, Av), u) + (B(v, v), Au) = 0 \tag{1.8}$$

We will also need several inequalities in 2-D (see refs. 6 and 7), one often referred to as Agmon's

$$\|u\|_\infty \leqslant c_1\, |u|^{1/2}\, |Au|^{1/2} \tag{1.9}$$

its alternative

$$\|u\|_\infty \leqslant c_2 \left(\ln \frac{|Au|}{\kappa_0\, \|u\|} + 1 \right)^{1/2} \|u\| \tag{1.10}$$

and finally, one known as Ladyzhenskaya's

$$|u|_{L^4(\Omega)} \leqslant c_3\, |u|^{1/2}\, \|u\|^{1/2} \tag{1.11}$$

Relations (1.9) and (1.10) are valid for any u in $\mathscr{D}_A$, while (1.11) holds in $\mathscr{D}_{A^{1/2}}$. Throughout the paper, constants c_i, $i = 1, 2, 3,\ldots$ are universal, and of the order of unity (uppercase C may have some dependencies, which will be noted).

If $u = u_{\kappa, 2\kappa}$ for some $\kappa > 0$, then (1.9) yields

$$\|u\|_\infty \leqslant c_1 \sqrt{2}\, \|u\| \tag{1.12}$$

The following result is used in Section 6.

Lemma 1.1. Let v, w be in V, with $P_\kappa w = w$. Then

$$|(B(v, v), w)| \leqslant c_4 \left(\ln \frac{\kappa}{\kappa_0} + 1 \right)^{1/2} \|v\|^2\, |w|, \qquad \text{with} \quad c_4 = \max\{2c_2, 12c_1\} \tag{1.13}$$

If moreover, $q \in V$ and $Q_\kappa q = q$, then

$$|(B(q, q), w)| \leqslant 12c_1\, |w| \sum_{n=1}^{\infty} \frac{\|r_n\|^2}{2^{2n}} \leqslant 3c_1\, \|q\|^2\, |w| \tag{1.14}$$

where

$$r_n = (P_{2^n\kappa} - P_{2^{n-1}\kappa})\, q$$

Proof. Write $v = p + q$, where $q = Q_\kappa v$, $p = P_\kappa v$, and $q' = P_{2\kappa} q$. Then since

$$P_\kappa B(p, Q_{2\kappa} v) = 0 = P_\kappa B(Q_{2\kappa} v, p)$$

we have

$$\begin{aligned}(B(v, v), w) &= (B(p, p), w) + (B(p, q), w) + (B(q, p), w) + (B(q, q), w)\\ &= (B(p, p), w) + (B(p, q'), w) + (B(q', p), w) + (B(q, q), w)\end{aligned}$$

and hence, immediately from (1.10) and (1.12)

$$|(B(p,p),w)|+|(B(p,q'),w)|+|(B(q',p),w)| \leqslant 2c_2\,\|p+q'\|^2\,|w|\left(\ln\frac{\kappa}{\kappa_0}+1\right)^{1/2}$$

Thus, (1.13) follows from (1.14).

To establish (1.14), first note that

$$P_{2^{n-1}\kappa}B(r_n,r_j)=0 \qquad \text{for} \quad j\neq n-1,n \tag{1.15}$$

so that by (1.4)

$$\begin{aligned}
|(B(q,q),w)| &\leqslant \sum_{\ell,n=1}^{\infty} |(B(r_\ell,r_n),w)| \\
&= \sum_{n=1}^{\infty} [|(B(r_n,r_n)+B(r_n,r_{n+1})+B(r_{n+1},r_n),w)|] \\
&\leqslant \sum_{n=1}^{\infty} [|(B(r_n,w),r_n)|+|(B(r_n,w),r_{n+1})|+|(B(r_{n+1},w),r_n)|] \\
&\leqslant \sum_{n=1}^{\infty} \|A^{1/2}w\|_\infty\,(|r_n|^2+2\,|r_n|\,|r_{n+1}|) \\
&\leqslant 3c_1\,|A^{1/2}w|^{1/2}\,|A^{3/2}w|^{1/2}\sum_{n=1}^{\infty}|r_n|^2 \\
&\leqslant 3c_1\,|w|\,\kappa^2\sum_{n=1}^{\infty}|r_n|^2 \\
&\leqslant 12c_1\,|w|\sum_{n=1}^{\infty}\frac{\|r_n\|^2}{2^{2n}} \qquad \blacksquare
\end{aligned}$$

We denote by S the solution operator defined by $S(t)\,u_0=u(t)$, where $u(t)$ is the unique solution to (1.1) such that $u(0)=u_0$. The global attractor $\mathscr{A}$ is defined by

$$\mathscr{A}=\bigcap_{t\geqslant 0} S(t)\,B$$

where B is a bounded absorbing set. Equivalently $\mathscr{A}$ is the largest bounded, invariant set (i.e., $S(t)\,\mathscr{A}=\mathscr{A}$ for all $t\geqslant 0$).

If we multiply (1.1) by u, (respectively Au) integrate over Ω and apply (1.4) and (1.5), we find that

$$\frac{1}{2}\frac{d}{dt}|u|^2+\nu\,\|u\|^2=(f,u) \tag{1.16}$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\nu\,|Au|^2=(f,Au) \tag{1.17}$$

In the scientific literature,

$$\frac{1}{L^2}|u|^2=2 \text{ times the total } \textit{energy} \text{ per unit mass} \tag{1.18}$$

and

$$\frac{1}{L^2}\|u\|^2=\text{the total } \textit{enstrophy} \text{ per unit mass} \tag{1.19}$$

See Remark A.1 for a derivation of (1.18) and (1.19). Relations (1.16) and (1.17) are the balance equations for the energy and enstrophy, respectively. Applying the Cauchy–Schwarz and Young inequalities to (1.16) gives

$$\frac{d}{dt}|u|^2+\nu\,\|u\|^2\leqslant\frac{|A^{-1/2}f|^2}{\nu}\leqslant\frac{|f|^2}{\nu\kappa_0^2} \tag{1.20}$$

from which we have

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|u\|^2\,d\tau\leqslant\frac{|f|^2}{\nu^2\kappa_0^2} \tag{1.21}$$

A similar procedure applied to (1.17) yields

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t|Au|^2\,d\tau\leqslant\frac{|f|^2}{\nu^2} \tag{1.22}$$

Applying a Gronwall inequality to (1.20) and its counterpart for the enstrophy gives the following well-known bounds on the attractor in V and H.

Lemma 1.2. For all $u\in\mathscr{A}$ we have $\|u\|\leqslant G\nu\kappa_0$ and $|u|\leqslant G_*\nu$, where

$$G=\frac{|f|}{\nu^2\kappa_0^2}\quad\text{and}\quad G_*=\frac{|A^{-1/2}f|}{\nu^2\kappa_0} \tag{1.23}$$

We will refer to G and G_* as the generalized and associated Grashof numbers, respectively. The former was introduced in ref. 9, while the latter was considered in ref. 10 only for the three-dimensional case. As will be seen in the next paragraphs, G_* is also useful in the two-dimensional case. We note that

$$G \leqslant \frac{\bar{\kappa}}{\kappa_0} G_*, \qquad \text{and} \qquad G_* \leqslant \frac{\kappa_0}{\underline{\kappa}} G \tag{1.24}$$

where $\underline{\kappa}$, $\bar{\kappa}$ are respectively, the largest and smallest wavenumbers such that

$$f = \sum_{\underline{\kappa} < |k| \, \kappa_0 \leqslant \bar{\kappa}} f_k e^{i\kappa_0 k \cdot x} \tag{1.25}$$

Note that this is equivalent to

$$P_{\underline{\kappa}} f = 0, \qquad Q_{\bar{\kappa}} f = 0 \tag{1.26}$$

Though we will occasionally note the specific dependence on $\underline{\kappa}$, $\bar{\kappa}$, we will assume throughout this paper that

$$\bar{\kappa}/\kappa_0 \leqslant C_0 \tag{1.27}$$

where C_0 is some fixed constant.

Another well-known, straight-forward estimate, this time using (1.11), on the difference of two solutions gives a lower bound on G in order to have a nontrivial attractor.

Proposition 1.3. For $\mathscr{A}$ to consist of more than a single element, we must have

$$G^{1/2} \geqslant \frac{1}{c_3} \tag{1.28}$$

where c_3 is as in (1.11).

2. STATIONARY STATISTICAL APPROACH

In fully developed turbulent flows, the physically interesting instantaneous quantities appear to behave unpredictably. However, time averages of those same quantities behave in a more predictable way; usually they can be reproduced in experiments. Therefore, the mathematical study of such flows should emphasize these averages. Yet for some continuous functions Φ and some initial data u_0, the limit, taken in the usual sense,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(\tau)\, u_0)\, d\tau \tag{2.1}$$

may not exist. To avoid this technical mathematical difficulty, we shall employ a generalized limit denoted $\mathrm{Lim}_{t \to \infty}$. This is defined as a linear functional on

$$B([0, \infty)) = \{g\colon g \text{ is a bounded real-valued function on } [0, \infty)\}$$

which satisfies

$$\mathop{\mathrm{Lim}}_{t \to \infty} g(t) = \lim_{t \to \infty} g(t)$$

whenever the right-hand side exists, and

$$\Big|\mathop{\mathrm{Lim}}_{t \to \infty} g(t)\Big| \leqslant \sup\{|g(t)|\colon 0 \leqslant t < \infty\} \qquad \text{for all} \quad g \in B([0, \infty))$$

The existence of such a linear functional is a direct consequence of the Hahn–Banach Theorem.[11] It follows that

$$\liminf_{t \to \infty} g(t) \leqslant \mathop{\mathrm{Lim}}_{t \to \infty} g(t) \leqslant \limsup_{t \to \infty} g(t) \qquad \text{for all} \quad g \in B([0, \infty)) \tag{2.2}$$

It is believed, but not yet proved, that in any experimental determination of the average of any $\Phi(S(t)\, u_0)$ the usual limit exists. The use of Lim makes this point moot.

We will use the following version of the Bogolyubov–Krylov theory (ref. 12), from ref. 13.

Proposition 2.1. For every $u_0 \in \mathscr{D}_A$ there exists an invariant probability measure μ_{u_0} such that

$$\mathop{\mathrm{Lim}}_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(\tau)\, u_0)\, d\tau = \int_{\mathscr{A}} \Phi(u)\, \mu_{u_0}(du) \tag{2.3}$$

for all real-valued continuous (with respect to the H-norm) functions Φ on $\mathscr{D}_A$.

From (1.21), (1.22) and (2.2) we can now easily infer that

$$\int_{\mathscr{A}} \|u\|^2 \,\mu_{u_0}(du) \leqslant \frac{|f|^2}{\nu^2 \kappa_0^2}, \qquad \int_{\mathscr{A}} |Au|^2 \,\mu_{u_0}(du) \leqslant \frac{|f|^2}{\nu^2} \tag{2.4}$$

for every $u_0 \in H$.

Let μ be an arbitrary invariant probability measure on $\mathscr{D}_A$. Then μ satisfies $\mu(\mathscr{A}) = 1$.[13] The next result allows us to infer that any estimate valid for all measures μ_{u_0} is also valid for any arbitrary invariant probability measure.

Lemma 2.2. For any invariant probability measure μ on $\mathscr{D}_A$

$$\int_{\mathscr{A}} \left[\int_{\mathscr{A}} \Phi(u)\, \mu_{u_0}(du) \right] \mu(du_0) = \int_{\mathscr{A}} \Phi(u_0)\, \mu(du_0) \tag{2.5}$$

Proof. By the Birkhoff Ergodic Theorem[11] there exists $\Phi^* = \Phi^*_\mu$ defined μ-a.e., such that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(\tau)\, u_0)\, d\tau = \Phi^*(u_0) \tag{2.6}$$

and

$$\int_{\mathscr{A}} \Phi^*(u)\, \mu(du) = \int_{\mathscr{A}} \Phi(u)\, \mu(du)$$

It follows that for an arbitrary invariant measure μ, the usual limit in (2.1) exists μ-a.e.. But the generalized limit in (2.3) agrees with the usual limit wherever the latter exists, and hence, in this case, $\Phi^{**} = \Phi^*$ μ-a.e., where

$$\Phi^{**}(u_0) = \operatorname*{Lim}_{t \to \infty} \frac{1}{t} \int_0^t \Phi(S(\tau)\, u_0)\, d\tau$$

Thus Φ^{**} is measurable with respect to μ and by (2.3)

$$\int_{\mathscr{A}} \left[\int_{\mathscr{A}} \Phi(u)\, \mu_{u_0}(du) \right] \mu(du_0) = \int_{\mathscr{A}} \Phi^{**}(u_0)\, \mu(du_0)$$

$$= \int_{\mathscr{A}} \Phi^*(u_0)\, \mu(du_0) = \int_{\mathscr{A}} \Phi(u_0)\, \mu(du_0) \quad \blacksquare$$

We remark that if μ is ergodic, then Φ^* in (2.6) is constant μ-a.e., but this is not necessary for what follows.

For a vector-valued Borel function $\Phi: H \to H$ such that

$$\int_{\mathscr{A}} |\Phi(u)| \, \mu(du) < \infty \tag{2.7}$$

one can define

$$\int_{\mathscr{A}} \Phi(u) \, \mu(du) \in H$$

as the unique vector in H satisfying

$$(v, w) = \int_{\mathscr{A}} (\Phi(u), w) \, \mu(du) \qquad \text{for all} \quad w \in H \tag{2.8}$$

With this definition, we readily obtain

$$\nu \int_{\mathscr{A}} Au\mu_{u_0}(du) + \int_{\mathscr{A}} B(u, u) \, \mu_{u_0}(du) = f \tag{2.9}$$

for all $u_0 \in H$. Taking a second average of each term in (2.9), but now with respect to any invariant probability measure μ, gives us by virtue of Lemma 2.2

$$\nu \int_{\mathscr{A}} Au\mu(du) + \int_{\mathscr{A}} B(u, u) \, \mu(du) = f \tag{2.10}$$

for any invariant probability measure μ. In order to simplify the notation we introduce the convention that once an invariant measure μ is chosen and held fixed in our consideration, then the averages, with respect to μ will be simply denoted as $\langle \cdot \rangle$, that is

$$\langle \Phi(u) \rangle = \int_{\mathscr{A}} \Phi(u) \, \mu(du) \tag{2.11}$$

Thus (2.10) can be written as

$$\nu \langle Au \rangle + \langle B(u, u) \rangle = f \tag{2.12}$$

Since by (2.4) we have

$$\int_{\mathscr{A}} |Au| \, \mu(du) \leqslant \left\{ \int_{\mathscr{A}} |Au|^2 \, \mu(du) \right\}^{1/2} < \infty$$

$\Phi(u) = Au$ satisfies (2.7), so that according to definition (2.8) we have

$$\langle (Au, A\langle u\rangle)\rangle = (\langle Au\rangle, A\langle u\rangle) \tag{2.13}$$

But obviously

$$\langle AP_\kappa u\rangle = A\langle P_\kappa u\rangle = AP_\kappa\langle u\rangle = P_\kappa A\langle u\rangle \to A\langle u\rangle$$

and thus

$$|\langle Au\rangle - A\langle P_\kappa u\rangle| = |\langle A(I-P_\kappa)\langle u\rangle|$$
$$\leqslant \langle |A(I-P_\kappa)\, u|\rangle \leqslant \left\{\int_{\mathscr{A}} |(I-P_\kappa)\, Au|^2\, \mu(du)\right\}^{1/2} \to 0$$

by Lebesgue's dominated convergence theorem. Thus

$$\langle Au\rangle = A\langle u\rangle \tag{2.14}$$

and we may rewrite (2.12) as

$$\nu A\langle u\rangle + \langle B(u,u)\rangle = f \tag{2.15}$$

This is the functional form of the Reynolds equation. Clearly, it can also be written as

$$\nu A\langle u\rangle + B(\langle u\rangle, \langle u\rangle) = f - \langle B(u-\langle u\rangle, u-\langle u\rangle)\rangle \tag{2.16}$$

where $\langle B(u-\langle u\rangle, u-\langle u\rangle)\rangle$ represents the contributions of the Reynolds stresses (associated to the present averaging) to the driving body force f. Explicitly denoting $w = u - \langle u\rangle$, we have

$$\langle B(u-\langle u\rangle, u-\langle u\rangle)\rangle = \mathscr{P}\left[\frac{\partial}{\partial x_i}\langle w_i w_j\rangle\right]_{j=1,2} \tag{2.17}$$

where the tensor in the square brackets is the Reynolds stress. (Compare (3.5) on p. 282 in ref. 2 with (5.1) on p. 260 of ref. 14.) We will supplement (2.15) with the following consequences of (1.16) and (1.17)

$$\langle \nu\, \|u\|^2\rangle = (f, \langle u\rangle) \tag{2.18}$$

$$\langle \nu\, |Au|^2\rangle = (f, A\langle u\rangle) \tag{2.19}$$

To obtain (2.18) first take the time average of (1.16) and deduce that

$$\int \nu \, \|u\|^2 \, \mu_{u_0}(du) = \int (f, u) \, \mu_{u_0}(du), \qquad \text{for all} \quad u_0 \in H$$

then apply Lemma 2.2. The proof for (2.19) uses (1.17) in a similar manner.

The relations (2.15), (2.18), and (2.19) have some interesting and useful consequences of their own. Taking the scalar product of (2.15) with respectively $A\langle u\rangle$, $\langle u\rangle$, $A^{-1}\langle u\rangle$, and applying respectively (2.19), (2.18), we obtain

$$\begin{aligned} &\nu \, |A\langle u\rangle|^2 + (\langle B\rangle, A\langle u\rangle) = (f, A\langle u\rangle) = \nu \langle |Au|^2\rangle \\ &\nu \, |A^{1/2}\langle u\rangle|^2 + (\langle B\rangle, \langle u\rangle) = (f, \langle u\rangle) = \nu \langle |A^{1/2}u|^2\rangle \\ &\nu \, |\langle u\rangle|^2 + (\langle B\rangle, A^{-1}\langle u\rangle) = (f, A^{-1}\langle u\rangle) \end{aligned} \tag{2.20}$$

where, for convenience, we have written simply B for $B(u, u)$.

From (2.13) we have that

$$\begin{aligned} \langle |A(u - \langle u\rangle)|^2\rangle &= \langle (Au, Au)\rangle - \langle (Au, A\langle u\rangle)\rangle - \langle (A\langle u\rangle, Au)\rangle \\ &\quad + \langle (A\langle u\rangle, A\langle u\rangle)\rangle \\ &= \langle |Au|^2\rangle - |A\langle u\rangle|^2 \end{aligned} \tag{2.21}$$

Similarly we have

$$\langle |A^{1/2}(u - \langle u\rangle)|^2\rangle = \langle |A^{1/2}u|^2\rangle - |A^{1/2}\langle u\rangle|^2 \tag{2.22}$$

Combining (2.21) and (2.22) with the first two equations in (2.20) gives

$$\begin{aligned} \nu(\langle B\rangle, A\langle u\rangle) &= \nu^2 \langle |A(u - \langle u\rangle)|^2\rangle \geqslant 0, \\ \nu(\langle B\rangle, \langle u\rangle) &= \nu^2 \langle |A^{1/2}(u - \langle u\rangle)|^2\rangle \geqslant 0 \end{aligned} \tag{2.23}$$

Now take the scalar product of (2.15) first with $\langle B\rangle$, $A^{-1}\langle B\rangle$

$$\begin{aligned} \nu(\langle B\rangle, A\langle u\rangle) + |\langle B\rangle|^2 &= (f, \langle B\rangle), \\ \nu(\langle B\rangle, \langle u\rangle) + |A^{-1/2}\langle B\rangle|^2 &= (f, A^{-1}\langle B\rangle) \end{aligned} \tag{2.24}$$

and secondly with f, $A^{-1}f$

$$\begin{aligned} \nu(f, A\langle u\rangle) + (\langle B\rangle, f) &= |f|^2, \\ \nu(f, \langle u\rangle) + (A^{-1}\langle B\rangle, f) &= |A^{-1/2}f|^2 \end{aligned} \tag{2.25}$$

Introducing (2.18), (2.19), (2.23), (2.24) into (2.25), we obtain

$$\begin{aligned} &\nu^2\langle|Au|^2\rangle+\nu^2\langle|A(u-\langle u\rangle)|^2\rangle+|\langle B\rangle|^2=|f|^2,\\ &\nu^2\langle|A^{1/2}u|^2\rangle+\nu^2\langle|A^{1/2}(u-\langle u\rangle)|^2\rangle+|A^{-1/2}\langle B\rangle|^2=|A^{-1/2}f|^2 \end{aligned} \tag{2.26}$$

or equivalently,

$$\begin{aligned} &2\nu^2\langle|Au|^2\rangle+|\langle B\rangle|^2=|f|^2+\nu^2\,|A\langle u\rangle|^2,\\ &2\nu^2\langle|A^{1/2}u|^2\rangle+|A^{-1/2}\langle B\rangle|^2=|A^{-1/2}f|^2+\nu^2\,|A^{1/2}\langle u\rangle|^2 \end{aligned} \tag{2.27}$$

Consequently

$$\begin{aligned} &\nu^2\langle|Au|^2\rangle+|\langle B\rangle|^2\leqslant|f|^2,\\ &\nu^2\langle|A^{1/2}u|^2\rangle+|A^{-1/2}\langle B\rangle|^2\leqslant|A^{-1/2}f|^2 \end{aligned} \tag{2.28}$$

We have from (2.15)

$$\nu AQ_{\bar\kappa}\langle u\rangle=-Q_{\bar\kappa}\langle B\rangle,\qquad \nu AP_{\underline\kappa}\langle u\rangle=-P_{\underline\kappa}\langle B\rangle \tag{2.29}$$

so that

$$\begin{aligned} \nu^2\,|A\langle u\rangle|^2-|\langle B\rangle|^2&=\nu^2(|AQ_{\bar\kappa}\langle u\rangle|^2+|A\langle u_{\underline\kappa,\bar\kappa}\rangle|^2+|AP_{\underline\kappa}\langle u\rangle|^2)\\ &\quad-(|Q_{\bar\kappa}\langle B\rangle|^2+|\langle B_{\underline\kappa,\bar\kappa}\rangle|^2+|P_{\underline\kappa}\langle B\rangle|^2)\\ &=\nu^2\,|A\langle u_{\underline\kappa,\bar\kappa}\rangle|^2-|\langle B_{\underline\kappa,\bar\kappa}\rangle|^2 \end{aligned}$$

and similarly

$$\nu^2\,|A^{1/2}\langle u\rangle|^2-|A^{-1/2}\langle B\rangle|^2=\nu^2\,|A^{1/2}\langle u_{\underline\kappa,\bar\kappa}\rangle|^2-|A^{-1/2}\langle B_{\underline\kappa,\bar\kappa}\rangle|^2$$

Thus (2.27) can be rewritten as

$$\begin{aligned} &2\nu^2\langle|Au|^2\rangle=|f|^2+\nu^2\,|A\langle u_{\underline\kappa,\bar\kappa}\rangle|^2-|\langle B_{\underline\kappa,\bar\kappa}\rangle|^2,\\ &2\nu^2\langle|A^{1/2}u|^2\rangle=|A^{-1/2}f|^2+\nu^2\,|A^{1/2}\langle u_{\underline\kappa,\bar\kappa}\rangle|^2-|A^{-1/2}\langle B_{\underline\kappa,\bar\kappa}\rangle|^2 \end{aligned} \tag{2.30}$$

Note that the vectors on the right hand sides of (2.30) belong to $(P_{\bar\kappa}-P_{\underline\kappa})\,H$. Our original proof of Theorem 7.7 was based on these relations. We preserve them here as they may be useful elsewhere.

3. AVERAGED ENSTROPHY AND ENERGY FLUXES

We start first with some mathematical results, valid in the framework considered in the previous section. In the next section, we shall discuss the relevance of those results for the Kraichnan theory of the inertial range of fully developed turbulent 2D fluid flows.

Let

$$p_\kappa = P_\kappa u \qquad \text{and} \qquad q_\kappa = Q_\kappa u$$

Multiply (1.1) by Ap_κ and use (1.5) and (1.7) to obtain

$$\frac{1}{2}\frac{d}{dt}\|p_\kappa\|^2 + \nu\,|Ap_\kappa|^2 = (B(p_\kappa, p_\kappa), Aq_\kappa) - (B(q_\kappa, q_\kappa), Ap_\kappa) + (f, Ap_\kappa)$$

$$= -L^2[\mathfrak{E}_\kappa^{\rightarrow} - \mathfrak{E}_\kappa^{\leftarrow}] + (f, Ap_\kappa) \tag{3.1}$$

where

$$\mathfrak{E}_\kappa^{\rightarrow}(u) = -\frac{1}{L^2}(B(p_\kappa, p_\kappa), Aq_\kappa) \qquad \text{and} \qquad \mathfrak{E}_\kappa^{\leftarrow}(u) = -\frac{1}{L^2}(B(q_\kappa, q_\kappa), Ap_\kappa) \tag{3.2}$$

are the *rates of enstrophy transfer* or *enstrophy fluxes* from low to high, and high to low wavenumbers, respectively, at wavenumber κ. Next, follow a similar procedure, except multiply by Aq_κ so that

$$\frac{1}{2}\frac{d}{dt}\|q_\kappa\|^2 + \nu\,|Aq_\kappa|^2 = L^2\mathfrak{E}_\kappa + (f, Aq_\kappa) = L^2[\mathfrak{E}_\kappa^{\rightarrow} - \mathfrak{E}_\kappa^{\leftarrow}] + (f, Aq_\kappa) \tag{3.3}$$

where the last equality defines $\mathfrak{E}_\kappa$, the *net rate of enstrophy transfer* (or *net enstrophy flux*) at the wavenumber κ; if $\mathfrak{E}_\kappa > 0$ ($\mathfrak{E}_\kappa < 0$ resp.), it means that this net transfer of enstrophy occurs from low to high (high to low) wavenumbers. Finally, repeat both procedures, only multiply by p_κ (respectively q_κ) before integrating to arrive at

$$\frac{1}{2}\frac{d}{dt}|p_\kappa|^2 + \nu\,\|p_\kappa\|^2 = -L^2\mathfrak{e}_\kappa + (f, p_\kappa)$$

$$\frac{1}{2}\frac{d}{dt}|q_\kappa|^2 + \nu\,\|q_\kappa\|^2 = L^2\mathfrak{e}_\kappa + (f, q_\kappa) \tag{3.4}$$

where $\mathfrak{e}_\kappa$ is the *net rate of energy transfer* (or *net energy flux*) at κ defined by

$$\mathfrak{e}_\kappa = \mathfrak{e}_\kappa^{\rightarrow} - \mathfrak{e}_\kappa^{\leftarrow} \tag{3.5}$$

with

$$\mathfrak{e}_\kappa^{\rightarrow}(u) = -\frac{1}{L^2}(B(p_\kappa, p_\kappa), q_\kappa) \quad \text{and} \quad \mathfrak{e}_\kappa^{\leftarrow}(u) = -\frac{1}{L^2}(B(q_\kappa, q_\kappa), p_\kappa)$$

Proposition 3.1. Suppose that $\kappa > \bar{\kappa}$. Then for any invariant probability measure μ

$$\langle \mathfrak{E}_\kappa(u) \rangle = \frac{\nu}{L^2} \langle |AQ_\kappa u|^2 \rangle \tag{3.6}$$

and

$$\langle \mathfrak{e}_\kappa(u) \rangle = \frac{\nu}{L^2} \langle \|Q_\kappa u\|^2 \rangle \tag{3.7}$$

Proof. Take the time average of (3.3) to obtain

$$\frac{1}{2t}(\|q_\kappa(t)\|^2 - \|q_\kappa(0)\|^2) + \frac{\nu}{t}\int_0^t |Aq_\kappa(\tau)|^2\, d\tau = \frac{L^2}{t}\int_0^t \mathfrak{E}_\kappa(u(\tau))\, d\tau$$

Since $\|q_\kappa\| \in B([0, \infty))$ it follows that

$$\operatorname*{Lim}_{t \to \infty} \frac{\nu}{t}\int_0^t |AQ_\kappa u(\tau)|^2\, d\tau = \operatorname*{Lim}_{t \to \infty} \frac{L^2}{t}\int_0^t \mathfrak{E}_\kappa(u(\tau))\, d\tau$$

Apply Proposition 2.1 to both sides to obtain

$$\nu \int_{\mathscr{A}} |AQ_\kappa u|^2\, \mu_{u_0}(du) = L^2 \int_{\mathscr{A}} \mathfrak{E}_\kappa(u)\, \mu_{u_0}(du)$$

The extension to any invariant measure now follows from (2.5). A similar procedure on (3.4) yields (3.7). ∎

Corollary 3.2. If $\kappa > \bar{\kappa}$, then for any invariant probability measure μ,

$$0 \leqslant \langle \mathfrak{e}_\kappa(u) \rangle \leqslant \frac{1}{\kappa^2} \langle \mathfrak{E}_\kappa(u) \rangle$$

Proposition 3.3. Suppose that $\kappa \leqslant \underline{\kappa}$. Then for any invariant probability measure μ

$$L^2 \langle \mathfrak{E}_\kappa(u) \rangle = -\nu \langle |AP_\kappa u|^2 \rangle \tag{3.8}$$

and

$$L^2 \langle \mathfrak{e}_\kappa(u) \rangle = -\nu \langle \|P_\kappa u\|^2 \rangle \tag{3.9}$$

Proof. Similar to that of Proposition 3.1. ∎

Note that (3.8) and (3.9) imply that any invariant measure depends on ν.

Corollary 3.4. If $\kappa \leqslant \underline{\kappa}$, and $\kappa > \kappa_0$, then

$$\kappa^2 \langle \mathfrak{e}_\kappa(u) \rangle \leqslant \langle \mathfrak{E}_\kappa(u) \rangle \leqslant 0 \tag{3.10}$$

Proof. Note that

$$-\kappa^2 L^2 \langle \mathfrak{e}_\kappa(u) \rangle = \nu\kappa^2 \langle \|P_\kappa u\|^2 \rangle \geqslant \nu \langle |AP_\kappa u|^2 \rangle = -L^2 \langle \mathfrak{E}_\kappa(u) \rangle \quad ∎$$

Thus while Corollary 3.2 shows that for large κ, the energy flux $\langle \mathfrak{e}_\kappa \rangle$ is negligible versus the enstrophy flux, Corollary 3.4 shows that for small κ it is the other way around.

Proposition 3.5. For $\kappa_1 \leqslant \kappa_2$ such that $(P_{\kappa_2} - P_{\kappa_1})\, f = 0$

$$\nu \langle |A(P_{\kappa_2} - P_{\kappa_1})\, u|^2 \rangle = L^2 \langle \mathfrak{E}_{\kappa_1}(u) \rangle - L^2 \langle \mathfrak{E}_{\kappa_2}(u) \rangle \tag{3.11}$$

Proof. Subtract (3.1) for $\kappa = \kappa_1$ from the same relation for $\kappa = \kappa_2$ and use

$$\|p_{\kappa_2}\|^2 = \|p_{\kappa_1}\|^2 + \|(P_{\kappa_2} - P_{\kappa_1})\, u\|^2, \qquad |Ap_{\kappa_2}|^2 = |Ap_{\kappa_1}|^2 + |A(P_{\kappa_2} - P_{\kappa_1})\, u|^2$$

we find that

$$\frac{1}{2}\frac{d}{dt} \|(P_{\kappa_2} - P_{\kappa_1})\, u\|^2 + \nu\, |A(P_{\kappa_2} - P_{\kappa_1})\, u|^2 = -L^2[\mathfrak{E}_{\kappa_2} - \mathfrak{E}_{\kappa_1}]$$

Applying Proposition 2.1 and Lemma 2.2 as before completes the proof. ∎

Remark 3.6. Observe that if

$$\nu\langle|A(P_{\kappa_2}-P_{\kappa_1})\,u|^2\rangle \ll L^2\langle\mathfrak{E}_{\kappa_1}(u)\rangle$$

then (3.11) yields

$$\langle\mathfrak{E}_{\kappa_2}(u)\rangle \approx \langle\mathfrak{E}_{\kappa_1}(u)\rangle \tag{3.12}$$

(See Section 4 for the physical interpretation of this result, and the appendix for the mathematical definitions of $\ll$ and $\approx$.)

In the scientific literature, the quantity $\langle\mathfrak{E}_\kappa(u)\rangle$ is referred to as the *enstrophy (per mass) flux at wavenumber* κ *according to the statistics given by the measure* μ.

Proposition 3.7. For all $\kappa \geqslant \kappa_0$

$$\langle|\mathfrak{E}_\kappa^{\leftarrow}(u)|\rangle \leqslant c_5 G\,\frac{\nu\kappa_0}{L^2\kappa}\,\langle|AQ_\kappa u|^2\rangle \overset{\text{if } \kappa\geqslant\bar{\kappa}}{=} c_5 G\,\frac{\kappa_0}{\kappa}\,\langle\mathfrak{E}_\kappa(u)\rangle \qquad \text{with} \quad c_5 = c_3^2 \tag{3.13}$$

Proof. From (1.11) it follows

$$L^2\,|\mathfrak{E}_\kappa^{\leftarrow}| \leqslant c_3^2\,|q_\kappa|^{1/2}\,\|q_\kappa\|^{3/2}\,|Ap_\kappa|^{1/2}\,\|Ap_\kappa\|^{1/2}$$

$$\leqslant c_5\,\|q_\kappa\|^2\,|Ap_\kappa| \leqslant \frac{c_5}{\kappa}\,|Aq_\kappa|^2\,\|p_\kappa\|$$

Using Lemma 1.2, we find that

$$L^2\,|\mathfrak{E}_\kappa^{\leftarrow}| \leqslant c_5\nu G\,\frac{\kappa_0}{\kappa}\,|Aq_\kappa|^2$$

Taking the time average yields the first relation in (3.13) while the second one follows from (3.6). ∎

Proposition 3.8. If $P_\kappa f = f$, then

$$\langle\mathfrak{E}_\kappa^{\rightarrow}(u)\rangle \leqslant \left(1 + c_5 G\,\frac{\kappa_0}{\kappa}\right)\langle\mathfrak{E}_\kappa(u)\rangle \tag{3.14}$$

and

$$\langle\mathfrak{E}_\kappa^{\leftarrow}(u)\rangle \leqslant \nu G\,\frac{\kappa_0}{\kappa}\left(1 + c_5 G\,\frac{\kappa_0}{\kappa}\right)^{-1}\langle\mathfrak{E}_\kappa^{\rightarrow}(u)\rangle \tag{3.15}$$

Proof. Clearly, (3.14) follows from (3.13) and the fact that $\mathfrak{E}_\kappa^{\rightarrow}(u) = \mathfrak{E}_\kappa(u) + \mathfrak{E}_\kappa^{\leftarrow}(u)$. It also follows from this fact and (3.13) that

$$\langle \mathfrak{E}_\kappa^{\leftarrow}(u)\rangle \leqslant c_5 G \frac{\kappa_0}{\kappa} (\langle \mathfrak{E}_\kappa^{\rightarrow}(u)\rangle - \langle \mathfrak{E}_\kappa^{\leftarrow}(u)\rangle)$$

Thus,

$$\left(1 + c_5 G \frac{\kappa_0}{\kappa}\right) \langle \mathfrak{E}_\kappa^{\leftarrow}(u)\rangle \leqslant c_5 G \frac{\kappa_0}{\kappa} \langle \mathfrak{E}_\kappa^{\rightarrow}(u)\rangle \quad \blacksquare$$

Proposition 3.9. If $P_\kappa f = f$ and

$$\kappa > c_5 \kappa_0 G \tag{3.16}$$

then

$$\langle |\mathfrak{E}_\kappa^{\leftarrow}(u)|\rangle \leqslant c_5 G \frac{\kappa_0}{\kappa} \left(1 - c_5 G \frac{\kappa_0}{\kappa}\right)^{-1} \langle \mathfrak{E}_\kappa^{\rightarrow}(u)\rangle \tag{3.17}$$

and

$$\langle \mathfrak{E}_\kappa^{\rightarrow}(u)\rangle \geqslant \frac{\nu}{L^2}\left(1 - c_5 G \frac{\kappa_0}{\kappa}\right) \langle |AQ_\kappa u|^2\rangle = \left(1 - c_5 G \frac{\kappa_0}{\kappa}\right) \langle \mathfrak{E}_\kappa(u)\rangle \tag{3.18}$$

Proof. By (3.13) it follows that

$$\langle |\mathfrak{E}_\kappa^{\leftarrow}|\rangle \leqslant c_5 G \frac{\kappa_0}{\kappa} \langle \mathfrak{E}_\kappa\rangle = c_5 G \frac{\kappa_0}{\kappa} [\langle \mathfrak{E}_\kappa^{\rightarrow}\rangle - \langle \mathfrak{E}_\kappa^{\leftarrow}\rangle] \leqslant c_5 G \frac{\kappa_0}{\kappa} [\langle \mathfrak{E}_\kappa^{\rightarrow}\rangle + \langle |\mathfrak{E}_\kappa^{\leftarrow}|\rangle]$$

from which (3.17) follows directly. Decompose the left hand side of (3.6) to obtain

$$L^2[\langle \mathfrak{E}_\kappa^{\rightarrow}\rangle - \langle \mathfrak{E}_\kappa^{\leftarrow}\rangle] = \nu \langle |AQ_\kappa u|^2\rangle$$

so that by (3.17) we have

$$\begin{aligned} L^2 \langle \mathfrak{E}_\kappa^{\rightarrow}\rangle &\geqslant \nu \langle |AQ_\kappa u|^2\rangle - L^2 \langle |\mathfrak{E}_\kappa^{\leftarrow}|\rangle \\ &\geqslant \nu \langle |AQ_\kappa u|^2\rangle - c_5 G \frac{\kappa_0}{\kappa}\left(1 - c_5 G \frac{\kappa_0}{\kappa}\right)^{-1} L^2 \langle \mathfrak{E}_\kappa^{\rightarrow}\rangle \end{aligned}$$

and consequently,

$$\left[1+c_5 G\frac{\kappa_0}{\kappa}\left(1-c_5 G\frac{\kappa_0}{\kappa}\right)^{-1}\right]\langle \mathfrak{E}_{\kappa}^{\rightarrow}\rangle \geqslant \frac{\nu}{L^2}\langle |AQ_\kappa u|^2\rangle$$

Multiplication by $(1-c_5 G\frac{\kappa_0}{\kappa})$ gives (3.18). ▮

4. KRAICHNAN THEORY

For obvious reasons, there are severe difficulties in carrying out experiments on turbulent flows in two dimensions. Consequently there are few direct empirical observations of such flows. Kraichnan's theory of fully developed turbulence in two dimensions[(1)] is largely dependent upon extrapolations from what is observed in 3-D, and on some reasonable physical-phenomenological arguments. Thus it is essential to establish the extent to which that theory is consistent with the properties of the Navier–Stokes equations in 2-D.

We note that the theory is based on the following

Empirical Assumptions 4.1

(a) At length scales much smaller than those of the enstrophy feeding structures, fully developed turbulence always looks the same.

(b) At the upper range (lowest wavenumbers) of the length scales in (a), the viscous dissipation of enstrophy is negligible and the motion is dominated by the transfer of enstrophy to smaller scales through a breakup of the eddies into smaller ones due to inertial effects.

(c) Most of the viscous dissipation of enstrophy takes place at the length scales in (a) which are much smaller than those in (b).

(d) The range in (c) is dominated by viscous effects.

(e) At the lower range (highest wavenumbers) of the length scales in (a), no significant relative movements occur.

In fact, a more specific mechanism on how inertial effects act in the range in (b) has been devised by Kraichnan,[(15)] namely

(b′) At the scales in (b), eddies break up into eddies of about half their linear size while traveling a distance comparable to their linear size. (For a more involved mechanism, reflecting intermittent events, see ref. 16.)

We recall in this section the famous heuristic inferences made from these assumptions by Kolmogorov,[(17)] Batchelor,[(18)] and Kraichnan.[(1)] We use this opportunity to give rigorous mathematical definitions based on the

Navier–Stokes equations for the physical quantities in those heuristic derivations. The heuristic theory may seem to be disconnected from the mathematical theory of the Navier–Stokes equations, but it is precisely this concern that this paper will address to some extent. Our mathematical setting is not identical to that of the conventional theory of fully developed turbulence; rather, it is more akin to that of numerical simulations of the latter. We hope that this section will introduce the heuristic theory of turbulence to a more mathematically oriented audience.

In the first part of this section, we will not make use of the specific mechanism in (b′); we will simply exploit the universality assumption in (a) to obtain the Kraichnan spectrum for two-dimensional turbulent flows, following Kolmogorov[17] and Batchelor.[19] After that derivation we will obtain the same result without resorting to the universality assumption and using instead the more precise mechanism in (b′).

The range in (b) is termed the *inertial range*, and that in (c), the *dissipation range*. The universality assumption (a) means that at those scales the statistical properties of the motion are independent of how the enstrophy is fed into the system. Moreover, it is presumed that the feeding occurs only at large length scales. According to (b) and (d), one basic physical quantity is the enstrophy. Since most of the enstrophy fed into the fluid is dissipated by viscosity within the dissipation range, one must also consider the viscosity and the rate of dissipation of enstrophy as basic physical quantities. Hence, in a more explicit formulation, the statistical quantities (related with a turbulent motion) should depend only on the length scale, the viscosity ν and the enstrophy dissipation rate

$$\eta \overset{\text{def}}{=} \frac{\nu}{L^2} \langle |Au|^2 \rangle \tag{4.1}$$

We now apply the universality assumption to deduce the form of the average energy in the eddies within a range of length scales. Namely, we consider

$$e_\kappa = 2 \text{ times the average energy per unit mass of the eddies}$$

$$\text{of linear size } \ell \in \left[\frac{1}{2\kappa}, \frac{1}{\kappa}\right), \qquad \text{that is} \quad \ell \sim \frac{1}{\kappa} \tag{4.2}$$

In terms of the solution to the Navier–Stokes equations e_κ, the average of (two times) the energy/mass over the modes with wavenumbers in $(\kappa, 2\kappa]$, is defined as

$$e_\kappa \overset{\text{def}}{=} \frac{1}{L^2} \langle |u_{\kappa, 2\kappa}|^2 \rangle \tag{4.3}$$

Within the range in (a), the quantity e_κ should vary with κ, but enjoy a universal property in the sense that it depends only on ν and η.

If we consider κ within the inertial subrange, then according to (b), the viscosity should play a minor role, so that e_κ should actually depend only on η and κ, say

$$e_\kappa = g(\eta, \kappa)$$

In particular, this relation should be independent of the choice of units for space and time. Thus, if we pass from x, t to $x' = \xi x$, $t' = \tau t$ we should still have

$$e'_{\kappa'} = g(\eta', \kappa')$$

From (4.3) we have

$$e'_{\kappa'} = \frac{\xi^2}{\tau^2} e_\kappa, \qquad \kappa' = \frac{\kappa}{\xi}, \qquad \eta' = \frac{\eta}{\tau^3}$$

that is

$$\frac{\xi^2}{\tau^2} g(\eta, \kappa) = g(\eta/\tau^3, \kappa/\xi)$$

Upon taking $\xi = \kappa$ and $\tau = \eta^{1/3}$ one obtains, within the inertial range,

$$e_\kappa = \frac{\tau^2}{\xi^2} g(1, 1) \sim \frac{\eta^{2/3}}{\kappa^2} \tag{4.4}$$

This relation is consistent with the classical estimate of the Kraichnan spectrum which is discussed in Remark 6.2.

We can also apply the universality assumption to obtain a wavenumber κ_{d} naturally associated with the dissipation range. According to the above, such a wavenumber should depend only on ν and η, say

$$\kappa_{\mathrm{d}} = h(\nu, \eta)$$

From the universality assumption, we obtain

$$\kappa'_{\mathrm{d}} = h(\nu', \eta')$$

From (4.1),

$$\kappa'_{\mathrm{d}} = \frac{\kappa_{\mathrm{d}}}{\xi}, \qquad \nu' = \frac{\xi^2 \nu}{\tau}, \qquad \eta' = \frac{\eta}{\tau^3}$$

that is

$$\frac{1}{\xi} h(\nu, \eta) = h(\xi^2 \tau^{-1} \nu, \tau^{-3} \eta)$$

With the choices

$$\frac{\xi^2}{\tau} = \frac{1}{\nu}, \qquad \tau^3 = \eta$$

we obtain

$$\kappa_{\mathrm{d}} = h(\nu, \eta) = \xi h(1, 1) \sim \left(\frac{\eta}{\nu^3}\right)^{1/6} \stackrel{\mathrm{def}}{=} \kappa_\eta \tag{4.5}$$

where the last equality defines κ_η. We note that at this moment, it is not clear whether most of the viscous enstrophy dissipation occurs around $\kappa_{\mathrm{d}} \sim \kappa_\eta$, or, for instance, beyond them.

We now present another derivation of (4.4) by Kraichnan, which uses the mechanism described in (b′) instead of the postulate (a). The mechanism in (b′) can be expressed in terms of the following quantities associated with a length scale $\ell \sim \kappa^{-1}$ as in (4.2),

U_κ = average velocity of eddies of size ℓ,

t_κ = average time for those eddies to travel the distance ℓ,

E_κ = average enstrophy per unit mass of eddies of linear size ℓ

Then,

$$U_\kappa \sim e_\kappa^{1/2}, \qquad t_\kappa \sim \frac{\ell}{U_\kappa} \sim \frac{1}{\kappa e_\kappa^{1/2}} \tag{4.6}$$

In terms of the solution to the Navier–Stokes equations, average of the enstrophy/mass, over the modes with wavenumbers in $(\kappa, 2\kappa]$, is defined by

$$E_\kappa \stackrel{\mathrm{def}}{=} \frac{1}{L^2} \langle |A^{1/2} u_{\kappa, 2\kappa}|^2 \rangle \tag{4.7}$$

It follows immediately from (4.7) that

$$E_\kappa \sim \kappa^2 e_\kappa \tag{4.8}$$

The feeding structures in the Navier–Stokes equations are embodied in the force f, so that the situation in (a) occurs for $\kappa \gg \bar{\kappa}$. At the length scales in (b), according to the eddy break up process in (b′), the *enstrophy flux per unit mass per unit time through wavenumber* κ,

$$\eta_\kappa \overset{\text{def}}{=} \frac{1}{L^2} \langle \mathfrak{E}_\kappa \rangle \tag{4.9}$$

accounts for most of the enstrophy dissipation of the eddies with linear size $\ell \sim \kappa^{-1}$ during the characteristic time t_κ and, hence, should satisfy

$$\eta_\kappa \sim \frac{E_\kappa}{t_\kappa} \sim \kappa^3 e_\kappa^{3/2}$$

i.e.,

$$e_\kappa \sim \frac{\eta_\kappa^{2/3}}{\kappa^2} \tag{4.10}$$

(For the consistency of this estimate with Kraichnan's form of the energy spectrum, see Remark 6.2.)

The rigorous definitions (4.7) and (4.9) for E_κ and η_κ yield

$$t_\kappa \sim \frac{\langle |A^{1/2} u_{\kappa,2\kappa}|^2 \rangle}{L^2 \langle \mathfrak{E}_\kappa \rangle} \tag{4.11}$$

The length scales in (b′) define a wavenumber interval $[\underline{\kappa}_i, \bar{\kappa}_i]$, the so-called inertial range. The *cascade of enstrophy* mechanism in (b′) means that

$$\eta_{\underline{\kappa}_i} \approx \eta_\kappa \approx \eta_{\bar{\kappa}_i} \tag{4.12}$$

for all $\underline{\kappa}_i \leqslant \kappa \leqslant \bar{\kappa}_i$.

According to (b) and (c), most of the enstrophy which is fed into the large wavenumbers is transferred through the inertial range only to be dissipated by viscosity in the dissipation range. Hence, the transfer of enstrophy per unit time to higher modes through a wavenumber within the inertial range should be the same as the enstrophy dissipation rate by viscous effects which occurs in the dissipation range. In other words,

$$\eta_\kappa \approx \eta \tag{4.13}$$

By (4.12) it follows that

$$e_\kappa \sim \frac{\eta^{2/3}}{\kappa^2}, \qquad \text{for} \quad \underline{\kappa}_i \leqslant \kappa \leqslant \bar{\kappa}_i \tag{4.14}$$

The time that it takes an eddy within the inertial range to break up into eddies with half their size is called the eddy *turnover time*. The cascade mechanism (b′) asserts that this time is of the order of the time that it takes an eddy to travel a length comparable to its linear size. We can also associate with such an eddy a *turnaround time*, which is the time that it takes an eddy to rotate once around its axis. It happens that the turnaround time is also of the same order of the turnover time. Indeed, the turnaround time of an eddy with linear size $\ell \sim 1/\kappa$ is approximately $1/E_\kappa^{1/2}$, which, according to (4.6) and (4.8), is equal to t_κ. We also note from (4.6) and (4.14) that the characteristic time t_κ is actually independent of κ,

$$t_\kappa \sim \frac{1}{\eta^{1/3}}, \qquad \text{for} \quad \underline{\kappa}_i \leqslant \kappa \leqslant \bar{\kappa}_i$$

Concerning (e), it can be described by

$$e_{\kappa'} \ll e_\kappa, \qquad \text{if} \quad \kappa \leqslant \kappa_\mathrm{d} \ll \kappa' \tag{4.15}$$

Condition (4.15) is the basis for the Landau–Lifschitz description of the number of degrees of freedom in turbulent flows. It suggests that the eddies of linear size much smaller than κ_d^{-1} are of no importance to the flow and need not be represented in a parameterization of the velocity field. Hence, we may retain only the eddies which can be resolved in a mesh of linear size of the order of κ_d^{-1}. Since there are approximately $(\kappa_\mathrm{d}/\kappa_0)^2$ squares with that linear size, the number of degrees of freedom in a turbulent flow should be of the order of $(\kappa_\mathrm{d}/\kappa_0)^2$.

In the conventional theory of turbulence, the form of the spectrum in the inertial range—(4.14) in the two-dimensional case—is usually assumed to hold up to the dissipation wavenumber κ_d. We will follow this convention here and assume that

$$e_\kappa \sim \frac{\eta^{2/3}}{\kappa^2}, \qquad \text{for} \quad \underline{\kappa}_i \leqslant \kappa \lesssim \kappa_\mathrm{d} \tag{4.16}$$

We choose κ_{d} as small as possible to still have

$$\nu \frac{1}{L^2} \langle |A(P_{\kappa_{\mathrm{d}}} - P_{\underline{\kappa}_i})|^2 \rangle \sim \eta \tag{4.17}$$

The eddies below $\underline{\kappa}_i$ were neglected in (4.17) since according to (c) there is nearly no enstrophy dissipation by viscosity at those scales, i.e.,

$$\nu \frac{1}{L^2} \langle |Au_{0, \underline{\kappa}_i}|^2 \rangle \ll \eta \tag{4.18}$$

From (4.17),

$$\eta \sim \nu \sum{}' \, 2^{4n} \kappa_{\mathrm{d}}^4 e_{2^n \kappa_{\mathrm{d}}}, \qquad \text{where} \quad \sum{}'$$

denotes the summation over all integers n such that $\underline{\kappa}_{\mathrm{i}} \leqslant 2^n \kappa_{\mathrm{d}} \leqslant \kappa_{\mathrm{d}}$. Using (4.16), we obtain

$$\begin{aligned} \eta &\sim \nu \kappa_{\mathrm{d}}^4 \sum{}' \, 2^{4n} \frac{\eta^{2/3}}{(2^n \kappa_{\mathrm{d}})^2} \\ &\sim \nu \eta^{2/3} \kappa_{\mathrm{d}}^2 \sum{}' \, 2^{2n} \\ &= \nu \eta^{2/3} \kappa_{\mathrm{d}}^2 \frac{1}{3} \left(4 - \left(\frac{\underline{\kappa}_i}{\kappa_{\mathrm{d}}} \right)^2 \right) \end{aligned}$$

and hence

$$\kappa_{\mathrm{d}} \sim \kappa_\eta \tag{4.19}$$

We recall that in ref. 10 it was rigorously proved that

$$\frac{1}{c_6} G^{1/6} \leqslant \frac{\kappa_\eta}{\kappa_0} \leqslant G^{1/3} \tag{4.20}$$

where the latter inequality also follows directly from (2.28) and the definitions of η, κ_η. It is easy to check that one can take $c_6 = (4c_3)^{1/6}$. The scenario in (a) requires that

$$\kappa_\eta \gg \bar{\kappa} \tag{4.21}$$

Since $\bar{\kappa}/\kappa_0$ is fixed, we have by (4.20) that a sufficient condition for (4.21) is

$$G^{1/6} \gg c_6 \tag{4.22}$$

If, on the other hand, (4.21) does hold, then dividing the end terms in (4.20) by $G^{1/6}$ gives us the necessary condition for (a)

$$G^{1/6} \gg 1/c_6 \tag{4.23}$$

Clearly, for either (4.22) and (4.23) to hold, we must have G large. Henceforth we will assume (4.23) holds.

We should emphasize here that, as will be proved in the next two sections (see Theorems 5.8 and 6.5 and Remark 6.7), $\bar{\kappa}_{\mathrm{i}}$ differs from κ_η by the factor $[\ln(\kappa_\eta/\underline{\kappa}_{\mathrm{i}})]^{-1/2}$, and that the enstrophy cascade begins at $\bar{\kappa}$, hence $\underline{\kappa}_{\mathrm{i}} \geqslant \bar{\kappa}$.

5. RIGOROUS RESULTS ON THE ENSTROPHY AND ENERGY CASCADE

Recall that we have proved that the mean enstrophy and energy fluxes are positive for $\kappa > \bar{\kappa}$, and negative for $\kappa \leqslant \underline{\kappa}$ (see Corollaries 3.2 and 3.4). In this section we measure the extent to which the cascades hold, and prove rigorously that the enstrophy cascade is more pronounced than the energy cascade in the small length scales, while the converse holds in the large length scales. By (4.9), we may express the enstrophy cascade (4.12) as

$$\langle \mathfrak{E}_\kappa \rangle \approx \langle \mathfrak{E}_{\kappa'} \rangle \qquad \text{in the enstrophy cascade range}$$

The energy cascade is then

$$\langle \mathfrak{e}_\kappa \rangle \approx \langle \mathfrak{e}_{\kappa'} \rangle \qquad \text{in the energy cascade range}$$

The accuracy of the cascade relations in (4.12) can be measured by

$$1 - \frac{\langle \mathfrak{E}_\kappa \rangle}{\langle \mathfrak{E}_{\kappa'} \rangle} \qquad \text{and} \qquad 1 - \frac{\langle \mathfrak{e}_\kappa \rangle}{\langle \mathfrak{e}_{\kappa'} \rangle} \tag{5.1}$$

The smaller the quantities in (5.1) are, the more accurate the corresponding cascade relations are. To compare their magnitudes we use the adimensional quotient

$$\frac{\kappa^2 \langle \mathfrak{e}_\kappa \rangle}{\langle \mathfrak{E}_\kappa \rangle}$$

Lemma 5.1. The following are equivalent

(i) $\langle \mathfrak{E}_{\bar{\kappa}} \rangle = 0$

(ii) $\langle \mathfrak{e}_{\bar{\kappa}} \rangle = 0$

(iii) $Q_{\bar{\kappa}} u = 0$, μ-a.e.

(iv) support of $\mu \subset P_{\bar{\kappa}} H$.

Proof. The first two relations are equivalent to the third by Proposition 3.1. Finally (iii) and (iv) are equivalent because of the continuity of the function $Q_\kappa u$. ∎

Lemma 5.2. The following are equivalent:

(i) $\langle \mathfrak{E}_{\underline{\kappa}} \rangle = 0$

(ii) $\langle \mathfrak{e}_{\underline{\kappa}} \rangle = 0$

(iii) $P_\kappa u = 0$, μ-a.e. for all $\kappa \leqslant \underline{\kappa}$

(iv) support of $\mu \subset Q_\kappa H$ for all $\kappa \leqslant \underline{\kappa}$.

Proof. Apply Proposition 3.3. ∎

Thus, provided the support of $\mu \not\subset P_{\bar{\kappa}} H$ (resp. supp $\mu \not\subset Q_\kappa H$ for all $\kappa \leqslant \underline{\kappa}$), we may define

$$\bar{\kappa}_\sigma = \left[\frac{\langle |AQ_{\bar{\kappa}} u|^2 \rangle}{\langle \|Q_{\bar{\kappa}} u\|^2 \rangle} \right]^{1/2}, \qquad \underline{\kappa}_\sigma = \left[\frac{\langle |AP_{\underline{\kappa}} u|^2 \rangle}{\langle \|P_{\underline{\kappa}} u\|^2 \rangle} \right]^{1/2}$$

Corollary 5.3. If $\bar{\kappa}_\sigma$ is defined, then $\bar{\kappa}_\sigma > \bar{\kappa}$; if $\underline{\kappa}_\sigma$ is defined, then $\underline{\kappa}_\sigma \leqslant \underline{\kappa}$.

Proof. If $\langle \|Q_{\bar{\kappa}}\|^2 \rangle > 0$, then since

$$|AQ_{\bar{\kappa}}|^2 \geqslant \lambda_{n+1} \, \|Q_{\bar{\kappa}} u\|^2$$

where $\lambda_{n+1} > \lambda_n = \bar{\kappa}^2$. The second assertion is obvious. ∎

Proposition 5.4. We have

$$0 \leqslant 1 - \frac{\langle \mathfrak{E}_\kappa \rangle}{\langle \mathfrak{E}_{\bar{\kappa}} \rangle} \leqslant \left(\frac{\kappa}{\bar{\kappa}_\sigma} \right)^2 \left[1 - \frac{\langle \mathfrak{e}_\kappa \rangle}{\langle \mathfrak{e}_{\bar{\kappa}} \rangle} \right], \qquad \text{for} \quad \bar{\kappa} \leqslant \kappa \leqslant \bar{\kappa}_\sigma \tag{5.2}$$

$$1 - \frac{\langle \mathfrak{E}_\kappa \rangle}{\langle \mathfrak{E}_{\underline{\kappa}} \rangle} \geqslant \left(\frac{\kappa}{\underline{\kappa}_\sigma} \right)^2 \left[1 - \frac{\langle \mathfrak{e}_\kappa \rangle}{\langle \mathfrak{e}_{\underline{\kappa}} \rangle} \right], \qquad \text{for} \quad \underline{\kappa}_\sigma \leqslant \kappa \leqslant \underline{\kappa} \tag{5.3}$$

Proof. Applying Proposition 3.1, noting that

$$|AQ_{\bar\kappa}u|^2 = |A(Q_{\bar\kappa} - Q_\kappa)\, u|^2 + |AQ_\kappa u|^2$$

(and similarly for $\|Q_{\bar\kappa}u\|^2$), we have

$$\begin{aligned}
1 - \frac{\langle \mathfrak{E}_\kappa \rangle}{\langle \mathfrak{E}_{\bar\kappa} \rangle} &= 1 - \frac{\langle |AQ_\kappa u|^2 \rangle}{\langle |AQ_{\bar\kappa} u|^2 \rangle} \\
&= \frac{\langle |A(Q_{\bar\kappa} - Q_\kappa)\, u|^2 \rangle}{\langle |AQ_{\bar\kappa} u|^2 \rangle} \\
&\leqslant \frac{\kappa^2 \langle \|(Q_{\bar\kappa} - Q_\kappa)\, u\|^2 \rangle}{\langle |AQ_{\bar\kappa} u|^2 \rangle} \\
&= \frac{\kappa^2}{\bar\kappa_\sigma^2} \frac{\langle \|(Q_{\bar\kappa} - Q_\kappa)\, u\|^2 \rangle}{\langle \|Q_{\bar\kappa} u\|^2 \rangle} \\
&\leqslant \left(\frac{\kappa}{\bar\kappa_\sigma}\right)^2 \left[1 - \frac{\langle \mathfrak{e}_\kappa \rangle}{\langle \mathfrak{e}_{\underline{\kappa}} \rangle}\right]
\end{aligned}$$

Relation (5.3) is proved in an analogous way using Proposition 3.3. ∎

Proposition 5.5. We have

$$\langle \mathfrak{E}_\kappa \rangle \geqslant \kappa^2 \langle \mathfrak{e}_\kappa \rangle \left[\left(\frac{\bar\kappa_\sigma}{\kappa}\right)^2 - 1\right], \qquad \text{for} \quad \bar\kappa \leqslant \kappa \leqslant \bar\kappa_\sigma \tag{5.4}$$

$$\kappa^2 \langle \mathfrak{e}_\kappa \rangle \leqslant \langle \mathfrak{E}_\kappa \rangle \left[1 - \left(\frac{\underline{\kappa}_\sigma}{\kappa}\right)^2\right], \qquad \text{for} \quad \underline{\kappa}_\sigma \leqslant \kappa \leqslant \underline{\kappa} \tag{5.5}$$

Proof. By Proposition 3.1, we have

$$\frac{\kappa^2 \langle \mathfrak{e}_\kappa \rangle}{\langle \mathfrak{E}_\kappa \rangle} = \frac{\kappa^2 \langle \|Q_\kappa u\|^2 \rangle}{\langle |AQ_\kappa u|^2} \leqslant \frac{\kappa^2 \langle \|Q_{\bar\kappa} u\|^2 \rangle}{\langle |AQ_{\bar\kappa} u|^2 \rangle \dfrac{\langle |AQ_\kappa u|^2 \rangle}{\langle |AQ_{\bar\kappa} u|^2 \rangle}} = \left(\frac{\kappa}{\bar\kappa_\sigma}\right)^2 \frac{\langle \mathfrak{E}_{\bar\kappa} \rangle}{\langle \mathfrak{E}_\kappa \rangle} \tag{5.6}$$

Since $\langle \mathfrak{e}_\kappa \rangle / \langle \mathfrak{e}_{\bar\kappa} \rangle > 0$, we have by (5.2)

$$1 - \frac{\langle \mathfrak{E}_\kappa \rangle}{\langle \mathfrak{E}_{\bar\kappa} \rangle} \leqslant \left(\frac{\kappa}{\bar\kappa_\sigma}\right)^2$$

which is equivalent to

$$\frac{\langle \mathfrak{E}_{\bar{\kappa}} \rangle}{\langle \mathfrak{E}_{\kappa} \rangle} \leqslant \frac{1}{1-(\kappa/\bar{\kappa}_\sigma)^2} \tag{5.7}$$

Using (5.7) in (5.6) gives us (5.4). The proof of (5.5) is similar. ∎

Remark 5.6

(i) Proposition 5.4 shows that for $\bar{\kappa} \leqslant \kappa \leqslant \bar{\kappa}_\sigma$, the enstrophy cascade is more expressed than the energy cascade, while for $\underline{\kappa}_\sigma \leqslant \kappa \leqslant \bar{\kappa}$, the opposite is true. These features are amplified if $\bar{\kappa}_\sigma \gg \bar{\kappa}$, $\underline{\kappa}_\sigma \ll \underline{\kappa}$, and κ is not too near $\bar{\kappa}_\sigma$, respectively $\underline{\kappa}_\sigma$.

(ii) Proposition 5.5 shows that if $\bar{\kappa}_\sigma \gg \kappa \geqslant \bar{\kappa}$, then the mean enstrophy flux dominates the dimensionally adjusted mean energy flux, $\kappa^2 \langle \mathfrak{e}_\kappa \rangle$. We consider these facts as a rigorous confirmation of Kraichnan's view that in 2-D turbulence the enstrophy cascade exists, and is more pronounced and more relevant than the energy cascade.

At this point we do not have a rigorous estimate for $\eta_\kappa = \langle \mathfrak{E}_\kappa \rangle$ in the cascade range $\bar{\kappa} \leqslant \kappa \ll \bar{\kappa}_\sigma$. Our next aim is to rigorously establish (4.13) on a certain interval contained in $[\bar{\kappa}, \bar{\kappa}_\sigma)$. For this let us recall that in turbulence theory the quantities η (see (4.1)) and

$$\epsilon = \frac{\nu}{L^2} \langle |A^{1/2} u|^2 \rangle \tag{5.8}$$

represent the averaged dissipation of enstrophy (resp. 2 times that of energy) per mass and per unit of time. Define the wavenumber

$$\kappa_\sigma = \left(\frac{\eta}{\epsilon}\right)^{1/2} = \left(\frac{\langle |Au|^2 \rangle}{\langle \|u\|^2 \rangle}\right)^{1/2} \tag{5.9}$$

It is easy to prove that $\kappa_\sigma \leqslant \bar{\kappa}_\sigma$ (provided the latter is defined) and that

$$\frac{\nu \langle |A u_{0,\kappa_{\mathrm{i}}}|^2 \rangle}{L^2 \eta} \leqslant \left(\frac{\kappa_{\mathrm{i}}}{\kappa_\sigma}\right)^2$$

and hence that the condition (4.18) is fulfilled if $\underline{\kappa}_{\mathrm{i}} \ll \kappa_\sigma$. Note also that

$$\nu \langle |AP_\kappa u|^2 \rangle \leqslant \nu \kappa^2 \langle |A^{1/2} P_\kappa u|^2 \rangle \leqslant \nu \kappa^2 \langle |A^{1/2} u|^2 \rangle \leqslant \nu \left(\frac{\kappa}{\kappa_\sigma}\right)^2 \langle |Au|^2 \rangle$$

and consequently, for $\kappa > \bar{\kappa}$, we have by Proposition 3.1 that

$$L^2 \langle \mathfrak{E}_\kappa \rangle = \nu \langle |AQ_\kappa u|^2 \rangle = \nu \langle |Au|^2 \rangle - \nu \langle |AP_\kappa u|^2 \rangle \geqslant \nu \langle |Au|^2 \rangle \left(1 - \left(\frac{\kappa}{\kappa_\sigma} \right)^2 \right) \tag{5.10}$$

Thus we have proved the following.

Lemma 5.7. For $\bar{\kappa} < \kappa$ we have

$$1 - \left(\frac{\kappa}{\kappa_\sigma} \right)^2 \leqslant \frac{\langle \mathfrak{E}_\kappa \rangle}{\eta} \leqslant 1 \tag{5.11}$$

Hence, given $\delta \in (0, 1]$, and $\kappa \in (\bar{\kappa}, \delta^{1/2} \kappa_\sigma]$ (which is possible only if $\kappa_\sigma / \bar{\kappa} \geqslant \delta^{-1/2}$), then due to (5.11)

$$\left| 1 - \frac{\langle \mathfrak{E}_\kappa \rangle}{\eta L^2} \right| \leqslant \delta$$

Clearly this means that if $\kappa_\sigma \gg \bar{\kappa}$ and

$$\bar{\kappa} < \kappa \ll \kappa_\sigma \tag{5.12}$$

then

$$\langle \mathfrak{E}_\kappa \rangle \approx \eta \tag{5.13}$$

The preceding can be summed up in the following.

Theorem 5.8. The interval in wavenumbers over which the enstrophy cascade conditions (4.12) and (4.13) holds contains $(\bar{\kappa}, C\kappa_\sigma)$, with C a small enough absolute constant, provided $\kappa_\sigma \gg \bar{\kappa}$.

Remark 5.9. If we introduce the wavenumber

$$\kappa_\tau = \frac{\langle \|u\|^2 \rangle}{\langle |u|^2 \rangle}$$

(for conescenti we notice that $1/\kappa_\tau \sim$ Taylor length) then the analog of Lemma 5.7 is that

$$1 - \left(\frac{\kappa}{\kappa_\tau} \right)^2 \leqslant \frac{\langle \mathfrak{e}_\kappa \rangle}{\epsilon} \leqslant 1, \qquad \text{for} \quad \kappa > \bar{\kappa}$$

while the analog of Theorem 5.8 is that if $\kappa_\tau \gg \bar{\kappa}$, then the energy cascade holds for κ satisfying $\bar{\kappa} < \kappa \ll \kappa_\tau$. The proofs are similar to those for Lemma 5.7 and Theorem 5.8.

Two significant relations between κ_τ and κ_σ are

$$\kappa_\tau \leqslant \kappa_\sigma \qquad \text{and} \qquad \kappa_\sigma \leqslant \bar{\kappa}_\sigma \left(1 - \frac{\bar{\kappa}^2}{\kappa_\tau^2}\right)^{1/2} \tag{5.14}$$

The first relation in (5.14) follows easily from

$$\langle \|u\|^2 \rangle \leqslant \langle |Au|\ |u| \rangle \leqslant \langle |Au|^2 \rangle^{1/2} \langle |u|^2 \rangle^{1/2}$$

while the second follows from

$$\kappa_\sigma^2 = \frac{\langle |AP_{\bar{\kappa}} u|^2 \rangle + \bar{\kappa}_\sigma^2 \langle \|Q_{\bar{\kappa}} u\|^2 \rangle}{\langle \|u\|^2 \rangle} = \frac{\langle |AP_{\bar{\kappa}} u|^2 \rangle - \bar{\kappa}_\sigma^2 \langle \|P_{\bar{\kappa}} u\|^2 \rangle}{\langle \|u\|^2 \rangle} + \kappa_\sigma^2 \geqslant \bar{\kappa}_\sigma^2 \left(1 - \frac{\bar{\kappa}^2}{\kappa_\tau^2}\right)$$

The relations in (5.14) provide additional support to Remark 5.6(ii).

Remark 5.10. We now have established the possibility of an interval range beyond $\bar{\kappa}$ in which (4.13) is valid. This provides a partial but rigorous confirmation of the basic assumption (b) in 4.1. However, in our approach the mathematical formulation of the eddy breakup mechanism in (b′) seems to be

$$\langle \mathfrak{E}_\kappa \rangle \approx \frac{1}{L^2} \langle -(B(u_{\kappa/2,\kappa}, u_{\kappa/2,\kappa}), Au_{\kappa,2\kappa}) \rangle \tag{5.15}$$

Our progress toward rigorously establishing (5.15) is noted in the following would-be derivation

$$\begin{aligned} \langle \mathfrak{E}_\kappa \rangle &\approx \langle \mathfrak{E}_\kappa^{\rightarrow} \rangle \\ &= \frac{1}{L^2} \langle -(B(u_{0,\kappa}, u_{0,\kappa}), Au_{\kappa,2\kappa}) \rangle \\ &\overset{?}{\approx} \frac{1}{L^2} \langle -(B(u_{\kappa/2,\kappa}, u_{\kappa/2,\kappa}), Au_{\kappa,2\kappa}) \rangle \end{aligned} \tag{5.16}$$

The first relation in (5.16) follows directly from Proposition 3.9 under the condition

$$\kappa \geqslant c_5 \kappa_0 G$$

which by (4.20) forces

$$\kappa \geqslant \kappa_\eta \left(\frac{\kappa_\eta}{\kappa_0}\right)^2$$

which in turn holds only deep in the dissipative range (see (4.19)). Thus two tasks remain for future research: extending Proposition 3.9 into the inertial range, and establishing the final $\approx$ relation in (5.16).

Remark 5.11. Assuming (5.16) holds, then one can rigorously prove that

$$e_{\kappa/2} + e_\kappa \geqslant (12 c_1 c_\kappa^{1/2})^{-2/3} \frac{\eta^{2/3}}{\kappa^2} \frac{1}{(\kappa L)^{2/3}} \tag{5.17}$$

where

$$c_\kappa = \frac{\langle |u_{\kappa/2,\,\kappa}|^4 \rangle}{\langle |u_{\kappa/2,\,\kappa}|^2 \rangle^2}$$

We omit the proof of (5.17) since the factor $(\kappa L)^{-2/3}$ makes the relation (5.17) much weaker than the relation (4.4) (see also (4.14)), which gives

$$e_{\kappa/2} + e_\kappa \sim \frac{\eta^{2/3}}{(\kappa/2)^2} + \frac{\eta^{2/3}}{\kappa^2} \sim \frac{\eta^{2/3}}{\kappa^2}$$

Remark 5.12. Although the Navier–Stokes equations have smoothing properties, and the Euler equations do not, for small viscosity, or equivalently, for large generalized Grashof numbers, the long time behavior of solutions to these equations bear similarities. Moreover, while the 2-D periodic Euler equations, forced at low wave numbers, do not develop arbitrarily small scales in finite time, they do display features in which the gradients of the vorticity are very large compared to the vorticity itself (see, for example, ref. 20). Good statistical estimates of these gradients and the vorticity itself are respectively η and ϵ. Therefore it is justified to consider the case where the ratio η/ϵ is large. One of the main rigorous results of this paper is that this condition is equivalent to the existence of the enstrophy cascade.

6. SUPPLEMENTAL RIGOROUS SUPPORT FOR THE KRAICHNAN THEORY

In Section 4 we heuristically inferred that an enstrophy cascade should hold for κ from some $\underline{\kappa}_{\rm i}$ up to some $\bar{\kappa}_{\rm i}$. In Section 5 we established that this is true at least up to a wavenumber comparable to κ_σ (albeit, much smaller). On the other hand, in Section 4, we also gave a heuristic argument that the dissipative range should start at a wavenumber comparable to κ_η. Hence from the heuristic point of view, $\kappa_\sigma \lesssim \kappa_\eta$. We start this section with a rigorous proof of this fact.

Theorem 6.1. If $G^{1/6} \gg c_6$, we have $\kappa_\sigma \leqslant C_1 \kappa_\eta$ where κ_η is defined in (4.5), and

$$C_1 = C_1\left(\frac{\bar{\kappa}}{\kappa_0}\right) = \left[\frac{\bar{\kappa}}{\kappa_0}\left(1 + c_4'\left(\ln\frac{\bar{\kappa}}{\kappa_0} + 1\right)^{1/2}\right)\right]^{1/3} \tag{6.1}$$

where $c_4' = 2\pi c_4$.

Proof. The relations (2.19) and (2.25) give

$$\nu\eta + \frac{1}{L^2}(\langle B(u, u)\rangle, f) = \frac{1}{L^2}|f|^2 \tag{6.2}$$

so by (1.13)

$$\frac{1}{L^2}|f|^2 \leqslant \nu\eta + c_4\left(\ln\frac{\bar{\kappa}}{\kappa_0} + 1\right)^{1/2}\frac{1}{L^2}\langle\|u\|^2\rangle\,|f| \tag{6.3}$$

Using (2.19) once again, along with definition (5.8), we have

$$\begin{aligned}\eta &= \frac{1}{L^2}\langle\nu\,|Au|^2\rangle = \frac{1}{L^2}(f, A\langle u\rangle)\\ &\leqslant \frac{1}{L^2}\|f\|\,\langle\|u\|^2\rangle^{1/2}\\ &\leqslant \frac{\bar{\kappa}}{L\nu^{1/2}}|f|\left\langle\nu\frac{1}{L^2}\|u\|^2\right\rangle^{1/2} = \frac{\bar{\kappa}}{L\nu^{1/2}}|f|\,\epsilon^{1/2}\end{aligned} \tag{6.4}$$

Apply (6.4) to (6.3) and use again (5.8) to obtain

$$\frac{1}{L^2}|f| \leqslant \nu^{1/2}\bar{\kappa}\frac{1}{L}\epsilon^{1/2} + \frac{c_4}{\nu}\left(\ln\frac{\bar{\kappa}}{\kappa_0} + 1\right)^{1/2}\epsilon \tag{6.5}$$

Introducing (6.5) in (6.4) gives

$$\eta \leqslant \bar{\kappa}^2\epsilon + c_4'\frac{\bar{\kappa}}{\kappa_0}\left(\ln\frac{\bar{\kappa}}{\kappa_0} + 1\right)^{1/2}\left(\frac{\epsilon}{\nu}\right)^{3/2} \tag{6.6}$$

If

$$\frac{\epsilon}{\nu^3} \leqslant \bar{\kappa}^2 \kappa_0^2 \tag{6.7}$$

then

$$\frac{\eta}{\nu^3} \leqslant \bar{\kappa}^2 \frac{\epsilon}{\nu^3} + c_4' \frac{\bar{\kappa}}{\kappa_0} \left(\ln \frac{\bar{\kappa}}{\kappa_0} + 1 \right)^{1/2} \left(\frac{\epsilon}{\nu^3} \right)^{3/2} \leqslant \bar{\kappa}^4 \kappa_0^2 \left[1 + c_4' \left(\ln \frac{\bar{\kappa}}{\kappa_0} + 1 \right)^{1/2} \right]$$

and hence (see (4.20))

$$G^{1/6} \lesssim \frac{\kappa_\eta}{\kappa_0} \leqslant \left(\frac{\bar{\kappa}}{\kappa_0} \right)^{2/3} \left[1 + c \left(\ln \frac{\bar{\kappa}}{\kappa_0} + 1 \right)^{1/2} \right]^{1/6}$$

which is inconsistent with (4.23). Hence we have the opposite of (6.7) which is equivalent to

$$\bar{\kappa} < \frac{1}{\kappa_0} \left(\frac{\epsilon}{\nu^3} \right)^{1/2} \tag{6.8}$$

Applying (6.8) to the first term on the right hand side of (6.6) yields

$$\eta^{2/3} \leqslant C_1^2 \frac{\epsilon}{\nu} \tag{6.9}$$

where C_1 is as in (6.1). Clearly (6.9) coincides with $\kappa_\sigma \leqslant C_1 \kappa_\eta$. ∎

Remark 6.2. In the physics and engineering literature dealing with turbulent flows it is commonly assumed that there exist time averages of the physically meaningful entities associated with the solution of the Navier–Stokes equations, and that these averages are independent of the initial data $u_0 \in \mathscr{A}$. In particular, it is taken for granted that the time average

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{1}{L^2} |A^\alpha (P_{\kappa'} - P_\kappa) S(\tau) u_0|^2 \, d\tau \tag{6.10}$$

exists and can be viewed, for $\kappa_0 \to 0$ as a Riemann sum for the integral in the wavenumbers

$$\int_\kappa^{\kappa'} \chi^{4\alpha} \mathscr{S}(\chi) \, d\chi \tag{6.11}$$

of some function $\mathscr{S}$, called the *energy spectrum* of the turbulent flow produced by f. The approximation of the integral (6.11) by the "Riemann sum" (6.10) should improve as $\kappa_0 \to 0$, or equivalently, as $L \to \infty$. In particular, e_κ is, for L large enough, a Riemann sum for

$$\int_\kappa^{2\kappa} \mathscr{S}(\chi)\, d\chi$$

Accepting the existence of the limit in (6.10) and its replacement by (6.11), Kraichnan's argument as presented in Section 4 (see (4.10)) shows that within the inertial range

$$\int_\kappa^{2\kappa} \mathscr{S}(\chi)\, d\chi \sim e_\kappa \overset{(4.10)}{\sim} \eta^{2/3}\kappa^{-2} \tag{6.12}$$

whence (assuming that (6.12) holds for $\underline{\kappa}_{\mathrm{i}} \leqslant \kappa \lesssim \kappa_{\mathrm{d}}$)

$$\int_\kappa^{\infty} \mathscr{S}(\chi)\, d\chi \sim \int_\kappa^{\kappa_{\mathrm{d}}} \eta^{2/3}\chi^{-3}\, d\chi \tag{6.13}$$

Note that

$$\mathscr{S}(\kappa) \sim \eta^{2/3}\kappa^{-3} \tag{6.14}$$

is consistent with (6.12) and (6.13). The relation (6.14) is the celebrated Kraichnan energy spectrum for 2-D turbulence. We consider a somewhat more explicit form of (6.14), and provide a rigorous basis for (6.13) in the following.

Proposition 6.3. Suppose that

$$\underline{\kappa}_{\mathrm{i}} \ll \kappa_\eta \tag{6.15}$$

and that

$$\mathscr{S}(\kappa) \approx c_{\mathrm{Kr}}\eta^{2/3}\kappa^{-3} \qquad \text{for} \quad \underline{\kappa}_{\mathrm{i}} \leqslant \kappa \leqslant \bar{\kappa}_{\mathrm{d}} \tag{6.16}$$

with

$$\bar{\kappa}_{\mathrm{d}} \overset{\mathrm{def}}{=} \left(\frac{2}{c_{\mathrm{Kr}}}\right)^{1/2} \kappa_\eta \tag{6.17}$$

and

$$c_{\mathrm{Kr}} \sim 1 \tag{6.18}$$

Then

$$\nu \frac{1}{L^2} \langle |Au_{\kappa,\infty}|^2 \rangle \ll \eta \qquad \text{for} \quad \kappa \geqslant \bar{\kappa}_{\mathrm{d}} \tag{6.19}$$

Proof. It suffices to show that

$$\nu \frac{1}{L^2} \langle |Au_{\underline{\kappa}_{\mathrm{i}},\bar{\kappa}_{\mathrm{d}}}|^2 \rangle \approx \eta \tag{6.20}$$

In turn, since

$$\nu \frac{1}{L^2} \langle |Au_{\underline{\kappa}_{\mathrm{i}},\bar{\kappa}_{\mathrm{d}}}|^2 \rangle \approx \nu \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} \chi^4 \mathscr{S}(\chi)\, d\chi$$

it suffices to show that

$$\nu \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} \chi^4 \mathscr{S}(\chi)\, d\chi \approx \nu c_{\mathrm{Kr}} \eta^{2/3} \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} \chi\, d\chi \approx \frac{1}{2} \nu c_{\mathrm{Kr}} \eta^{2/3} \bar{\kappa}_{\mathrm{d}}^2 = \eta \tag{6.21}$$

The last relation in (6.21) follows immediately from the definition in (6.17). To prove the second relation in (6.21), we write

$$\begin{aligned}
\beta \stackrel{\text{def}}{=} \nu c_{\mathrm{Kr}} \eta^{2/3} \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} \chi\, d\chi &= \frac{\nu c_{\mathrm{Kr}}}{2} \eta^{2/3} [\bar{\kappa}_{\mathrm{d}}^2 - \underline{\kappa}_{\mathrm{i}}^2] \\
&= \frac{\nu c_{\mathrm{Kr}}}{2} \eta^{2/3} \left[\frac{2\kappa_\eta^2}{c_{\mathrm{Kr}}} - \underline{\kappa}_{\mathrm{i}}^2 \right] \\
&= \eta \left(\frac{\nu}{\eta^{1/3}} \right) \left[\kappa_\eta^2 - \frac{c_{\mathrm{Kr}}}{2} \underline{\kappa}_{\mathrm{i}}^2 \right] \\
&= \eta \left[1 - \frac{c_{\mathrm{Kr}}}{2} \left(\frac{\underline{\kappa}_{\mathrm{i}}}{\kappa_\eta} \right)^2 \right]
\end{aligned}$$

and observe that by (6.18) and (6.15)

$$\left|1-\frac{\beta}{\eta}\right| = \frac{c_{\mathrm{Kr}}}{2}\left(\frac{\underline{\kappa}_{\mathrm{i}}}{\kappa_\eta}\right)^2 \ll 1$$

To prove the first relation in (6.21) we write

$$\begin{aligned}\alpha \stackrel{\text{def}}{=} \nu \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} \chi^4 \mathscr{S}(\chi)\, d\chi &= \nu \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} \chi^4 \left(\frac{\mathscr{S}(\chi)}{c_{\mathrm{Kr}}\eta^{2/3}\chi^{-3}}\right) c_{\mathrm{Kr}}\eta^{2/3}\chi^{-3}\, d\chi \\ &= \beta + \nu \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} c_{\mathrm{Kr}}\eta^{2/3}\chi \left(\frac{\mathscr{S}(\chi)}{c_{\mathrm{Kr}}\eta^{2/3}\chi^{-3}} - 1\right) d\chi\end{aligned}$$

Applying (6.16) yields

$$\left|1-\frac{\alpha}{\beta}\right| \leqslant \sup_{\underline{\kappa}_{\mathrm{i}} \leqslant \kappa \leqslant \bar{\kappa}_{\mathrm{d}}} \left|1 - \frac{\mathscr{S}(\chi)}{c_{\mathrm{Kr}}\eta^{2/3}\chi^{-3}}\right| \ll 1 \quad ∎$$

Corollary 6.4. Under the assumptions of Proposition 6.3 we have that

$$\frac{\nu}{L^2}\langle |Au_{\kappa,\bar{\kappa}_{\mathrm{d}}}|^2\rangle \approx \eta \qquad \text{for all} \quad \kappa \ll \bar{\kappa}_{\mathrm{d}}$$

Proof. Observe that

$$\begin{aligned}\frac{\nu}{L^2}\langle |Au_{\kappa,\bar{\kappa}_{\mathrm{d}}}|^2\rangle &\approx \nu \int_{\kappa}^{\bar{\kappa}_{\mathrm{d}}} \chi^4 \mathscr{S}(\chi)\, d\chi \\ &\approx \nu c_{\mathrm{Kr}}\eta^{2/3} \int_{\kappa}^{\bar{\kappa}_{\mathrm{d}}} \chi\, d\chi \\ &= \frac{\nu c_{\mathrm{Kr}}}{2}\eta^{2/3}[\bar{\kappa}_{\mathrm{d}}^2 - \kappa^2] = \eta\left[1-\left(\frac{\kappa}{\bar{\kappa}_{\mathrm{d}}}\right)^2\right] \quad ∎\end{aligned}$$

Theorem 6.5. Under the assumptions of Proposition 6.3 we have that

$$\eta \approx \nu \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} \kappa^4 \frac{\eta^{2/3}}{\kappa^3}\, d\kappa \approx \frac{\nu}{2}\eta^{2/3}\bar{\kappa}_{\mathrm{d}}^2 \tag{6.22}$$

$$\kappa_\eta \sim \bar{\kappa}_{\mathrm{d}} \tag{6.23}$$

$$\kappa_\eta^2 \sim \kappa_\sigma^2 \ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}} \tag{6.24}$$

and

$$\bar{\kappa} \leqslant \underline{\kappa}_{\mathrm{i}} \ll \kappa_\sigma \leqslant \kappa_\eta \tag{6.25}$$

Proof. The first relation in (6.22) follows from (6.20), the second follows from (6.15), and (6.23) follows immediately from the definition of κ_η in (4.5). By (6.19) we have

$$\frac{\nu}{L^2}\langle \|Q_\kappa u\|^2\rangle \ll \frac{\eta}{\kappa^2} \leqslant \frac{\eta}{\bar{\kappa}_{\mathrm{d}}^2} \approx \frac{\nu}{2}\eta^{2/3} \qquad \text{for} \quad \kappa \geqslant \bar{\kappa}_{\mathrm{d}}$$

We write

$$\frac{\nu}{L^2}\langle \|u\|^2\rangle = \frac{\nu}{L^2}\langle \|P_{\bar{\kappa}_{\mathrm{d}}} u\|^2\rangle + \frac{\nu}{L^2}\langle \|Q_{\bar{\kappa}_{\mathrm{d}}} u\|^2\rangle$$

and note that by (4.18) and (6.23)

$$\frac{\nu}{L^2}\langle \|P_{\bar{\kappa}_{\mathrm{d}}} u\|^2\rangle \sim \nu \int_{\underline{\kappa}_{\mathrm{i}}}^{\bar{\kappa}_{\mathrm{d}}} \frac{\eta^{2/3}}{\chi}\, d\chi$$

$$= \nu\eta^{2/3} \ln \frac{\bar{\kappa}_{\mathrm{d}}}{\underline{\kappa}_{\mathrm{i}}} \sim \nu\eta^{2/3} \ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}}$$

whence

$$\epsilon = \frac{\nu}{L^2}\langle \|u\|^2\rangle \sim \nu\eta^{2/3} \ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}} \tag{6.26}$$

Multiplying (6.26) by $\eta^{1/3}(\nu\epsilon)^{-1}$ we have

$$\kappa_\eta^2 = \frac{\eta^{1/3}}{\nu} \sim \frac{\eta}{\epsilon} \ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}} = \kappa_\sigma^2 \ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}} \tag{6.27}$$

The second relation in (6.25) follows from (6.15). ∎

Remark. If the Kraichnan theory holds for the invariant measure μ, it is necessary that $\kappa_\sigma/\bar{\kappa} \gg 1$.

As an aside, note that the role of the constant C_1 in (6.9) is played by $(\ln(\kappa_\eta/\underline{\kappa}_{\mathrm{i}}))^{-1}$ in (6.27), making the latter the stronger relation as κ_η increases.

This should be complemented with the following.

Remark 6.6.

$$\nu \frac{1}{L^2} \langle |A(P_{\kappa_\eta} - P_{\kappa_\sigma})\, u|^2 \rangle = \nu \int_{\kappa_\sigma}^{\kappa_\eta} \kappa^4 \frac{\eta^{2/3}}{\kappa^3}\, d\kappa$$

$$= \frac{\nu}{2} \eta^{2/3} (\kappa_\eta^2 - \kappa_\sigma^2)$$

$$\sim \frac{\nu}{2} \eta^{2/3} \frac{\eta^{1/3}}{\nu} \left(1 - \frac{1}{\ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}}}\right)$$

$$= \frac{1}{2} \eta \left(1 - \frac{1}{\ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}}}\right)$$

so

$$\nu \frac{1}{L^2} \langle |A(P_{\kappa_\eta} - P_{\kappa_\sigma})\, u|^2 \rangle \sim \eta$$

In other words, strictly speaking, if (6.14) holds up to κ_η, then the interval $[\kappa_\sigma, \kappa_\eta]$ is beyond the enstrophy cascade range.

Remark 6.7. For the two-dimensional case we consider in this paper, the counterpart of the well known Kolmogorov relation

$$\epsilon \sim \frac{U^3}{L}$$

where

$$U = \left(\frac{\langle |u|^2 \rangle}{L^2} \right)^{1/2}$$

is

$$\eta \sim \frac{U^3}{L^3} \tag{6.28}$$

The inequality

$$\epsilon \leqslant C \frac{U^3}{L}$$

was rigorously established first for the case of channel flow in refs. 21 and 22 (see also ref. 23 for the periodic case).

Theorem 6.8. We have

$$\eta \leqslant C_2 \frac{U^3}{L^3} \tag{6.29}$$

where

$$C_2 = (2\pi C_0)^4 (1 + c_1) \tag{6.30}$$

Proof. From (6.2) and (1.9) we have

$$\begin{aligned} \frac{1}{L^2} |f|^2 &\leqslant \nu\eta + c_1 \frac{1}{L^2} \langle |u|^2 |A^{1/2} f|^{1/2} |A^{3/2} f|^{1/2} \rangle \\ &\leqslant \nu\eta + c_1 \frac{1}{L^2} \langle |u|^2 \rangle \bar{\kappa}^2 |f| = \nu\eta + c_1 U^2 \bar{\kappa}^2 |f| \end{aligned} \tag{6.31}$$

From (2.19) we have that

$$\eta = \frac{1}{L^2} \langle \nu |Au|^2 \rangle = \frac{1}{L^2} (f, A\langle u \rangle) \leqslant \frac{1}{L^2} |\langle u \rangle| |Af| \leqslant \frac{U}{L} \bar{\kappa}^2 |f| \tag{6.32}$$

Using this in (6.31) gives

$$\frac{1}{L^2} |f| \leqslant \nu U \bar{\kappa}^2 \frac{1}{L} + c_1 U^2 \bar{\kappa}^2 \tag{6.33}$$

If $\frac{\nu}{L} \geqslant U$, then

$$\frac{1}{L^2} |f| \leqslant \nu^2 \bar{\kappa}^2 \frac{1}{L^2} (1 + c_1)$$

whence

$$G \leqslant (1 + c_1) \left(\frac{\bar{\kappa}}{\kappa_0} \right)^2$$

which does not allow (4.23) to hold. Consequently we can assume that $\frac{\nu}{L} < U$, hence

$$\frac{1}{L}|f| \leqslant (1+c_1)\, U^2 \bar{\kappa}^2 L \tag{6.34}$$

Introducing this in (6.32) we obtain

$$\eta \leqslant (1+c_1)\, U^3 \bar{\kappa}^4 L = (1+c_1)(\bar{\kappa} L)^4 \left(\frac{U}{L}\right)^3$$

which clearly implies (6.29) and (6.30). ▮

Remark 6.9. If the Kraichnan theory holds, then using (4.16) we have

$$U^2 = \langle |u|^2 \rangle \frac{1}{L^2} \geqslant \sum_{\underline{\kappa}_{\mathrm{i}} \leqslant 2^n \kappa_\eta \leqslant \kappa_\eta} e_{2^n \kappa_\eta}$$
$$\sim \sum_{\underline{\kappa}_{\mathrm{i}} \leqslant 2^n \kappa_\eta \leqslant \kappa_\eta} \frac{\eta^{2/3}}{(2^n \kappa_\eta)^2} = \eta^{2/3} \frac{4}{3} \left(\frac{1}{\underline{\kappa}_{\mathrm{i}}^2} - \frac{1}{4\kappa_\eta^2} \right)$$

so that

$$U^2 \gtrsim \eta^{2/3} \frac{1}{\underline{\kappa}_{\mathrm{i}}^2} \tag{6.35}$$

However, notice also that the rigorous estimate (6.29) can be written as

$$\eta^{2/3} \lesssim U^2 \kappa_0^2 \sim U^2 \bar{\kappa}^2$$

so that the following is a *rigorous* estimate

$$U^2 \gtrsim \frac{\eta^{2/3}}{\bar{\kappa}^2} \geqslant \frac{\eta^{2/3}}{\underline{\kappa}_{\mathrm{i}}^2} \tag{6.36}$$

Comparing (6.35) with (6.36), we see that the latter is a rigorous, albeit severely limited, confirmation of the Kraichnan energy spectrum. Indeed we have the following relation

$$U^2 \sim \int_{\underline{\kappa}_{\mathrm{i}}}^{\kappa_{\mathrm{d}}} \mathscr{S}(\chi)\, d\chi \sim \int_{\underline{\kappa}_{\mathrm{i}}}^{\kappa_{\mathrm{d}}} \frac{\eta^{2/3}}{\chi^3}\, d\chi = \frac{\eta^{2/3}}{2} \left(\frac{1}{\underline{\kappa}_{\mathrm{i}}^2} - \frac{1}{\kappa_{\mathrm{d}}^2} \right) \sim \frac{\eta^{2/3}}{\underline{\kappa}_{\mathrm{i}}^2}$$

Remark 6.10. For the benefit of the reader we note that although we do not consider the log correction to the Kraichnan spectrum (see refs. 24 and 25) this will not change the conclusion of our estimates, because the rigorous estimates we obtain are compatible with the log corrected estimates. Our rigorous results are not sharp enough to detect the presence of the log correction.

7. NECESSARY CONDITIONS FOR THE VALIDITY OF THE KRAICHNAN THEORY

This section isolates a set of assumptions, both purely mathematical, and heuristic, under which the results above connect to the Kraichnan theory of fully developed two-dimensional turbulence. Let μ be an invariant probability measure on $\mathscr{A}$ that shall remain fixed throughout this section.

To be more precise, we will say that the *Kraichnan theory of turbulence* holds, if

[K1] $\nu L^{-2}\langle |Au_{\kappa,\infty}|^2\rangle \ll \eta$ for $\kappa \gg \kappa_\eta$,

[K2] The cascade condition (4.12) holds for $\underline{\kappa}_{\mathrm{i}} \leqslant \kappa \leqslant \bar{\kappa}_{\mathrm{i}}$ with $\underline{\kappa}_{\mathrm{i}} \ll \bar{\kappa}_{\mathrm{i}}$,

[K3] The relations (6.15), (6.16) hold with $\bar{\kappa}_{\mathrm{d}}$ and c_{Kr} as in (6.17) and (6.18) respectively.

Proposition 7.1

(i) [K3] implies [K1] and [K2] (with $\bar{\kappa}_{\mathrm{i}} \sim \bar{\kappa}_{\mathrm{d}}$),

(ii) [K3] implies

$$\bar{\kappa} \ll \kappa_\sigma \tag{7.1}$$

(iii) (7.1) implies [K2] and (6.15)

Proof. To prove (i), note that [K1] is immediate and use Corollary 5.8 and the analog of (5.10) to establish [K2]. By (1.27) we have that (6.15) is equivalent to $\bar{\kappa} \ll \kappa_\eta$. From (6.27) we have

$$\frac{\kappa_\eta}{\bar{\kappa}} \sim \frac{\kappa_\sigma}{\bar{\kappa}}\left(\ln \frac{\kappa_\eta}{\bar{\kappa}}\right)^{1/2}$$

Set

$$\alpha = \frac{\kappa_\eta}{\bar{\kappa}} \qquad \text{and} \qquad \beta = \frac{\kappa_\sigma}{\bar{\kappa}}$$

There exists constants c_9, c_{10} such that

$$c_9\alpha(\ln \alpha)^{-1/2} < \beta < c_{10}\alpha(\ln \alpha)^{-1/2}$$

from which it is clear that $\alpha \gg 1$ if and only if $\beta \gg 1$. That (7.1) implies [K2] follows from 5.8; that it implies (6.15) follows from 6.1. ∎

Remark 7.2. Note that $\kappa_\sigma \gg \bar{\kappa}$ implies the existence of an inertial range, whereas the Kraichnan theory implies $\kappa_\sigma \gg \bar{\kappa}$.

We now derive explicit conditions on both the force f and the invariant measure μ that are equivalent to (7.1). First, we need to define some quantities which were used to prove in ref. 26, that for the Kraichnan theory to hold, the force f must have at least two modes with distinct wavenumbers.

To start, for any v in L^2 let v_κ denote the L^2 projection of v onto the eigenspace of A associated with κ. In particular then, we have

$$f = \sum_{\underline{\kappa} \leqslant \kappa \leqslant \bar{\kappa}} f_\kappa \qquad \text{and} \qquad \langle u \rangle = \sum_{\kappa_0 \leqslant \kappa} \langle u \rangle_\kappa$$

and $(Af)_\kappa = \kappa^2 f_\kappa$, $(A\langle u \rangle)_\kappa = \kappa^2 \langle u \rangle_\kappa$. It follows from (2.19) that

$$\begin{aligned} \langle \nu\, |Au|^2 \rangle &= (f, A\langle u \rangle) = \sum \kappa^2 (f_\kappa, \langle u \rangle_\kappa) \\ &= \sum \kappa^2 (f_\kappa, \langle u \rangle_\kappa)^+ - \sum \kappa^2 (f_\kappa, \langle u \rangle_\kappa)^- \end{aligned} \tag{7.2}$$

and similarly

$$\begin{aligned} \nu \langle \|u\|^2 \rangle &= (f, \langle u \rangle) = \sum (f_\kappa, \langle u \rangle_\kappa) \\ &= \sum (f_\kappa, \langle u \rangle_\kappa)^+ - \sum (f_\kappa, \langle u \rangle_\kappa)^- \end{aligned} \tag{7.3}$$

where for a real number a

$$a^+ = \max\{a, 0\} \qquad \text{and} \qquad a^- = -\min\{a, 0\}$$

and the summation in both relations, as well as throughout the remainder of this section, is taken over $\underline{\kappa} \leqslant \kappa \leqslant \bar{\kappa}$. We rewrite (7.3) as

$$(f, \langle u \rangle) + r_- = r_+ \tag{7.4}$$

where

$$r_+ = \sum (f_\kappa, \langle u \rangle_\kappa)^+ \qquad \text{and} \qquad r_- = \sum (f_\kappa, \langle u \rangle_\kappa)^-$$

Suppose that $(f, \langle u \rangle) = 0$, so that by (7.3) we have $\nu \langle \|u\|^2 \rangle = 0$. It follows that $u = 0$ a.e.-μ, so that taking the average of (1.1), we find that $f = 0$. But we consider only $G > 0$. So, in fact $r_+ \geqslant (f, \langle u \rangle) > 0$, and thus we may write

$$\langle \nu \, |Au|^2 \rangle = \left[\frac{\sum \kappa^2 (f_\kappa, \langle u \rangle_\kappa)}{\sum (f_\kappa, \langle u \rangle_\kappa)} \right] \sum (f_\kappa, \langle u \rangle_\kappa) = \kappa_\sigma^2 \nu \langle \|u\|^2 \rangle \tag{7.5}$$

If $r_- = 0$, then

$$\kappa_\sigma^2 = \frac{\sum \kappa^2 (f_\kappa, \langle u \rangle_\kappa)}{\sum (f_\kappa, \langle u \rangle_\kappa)^+} \leqslant \frac{\sum \kappa^2 (f_\kappa, \langle u \rangle_\kappa)^+}{\sum (f_\kappa, \langle u \rangle_\kappa)^+} \leqslant \bar{\kappa}^2$$

Thus merely asking that $\kappa_\sigma > \bar{\kappa}$ (let alone that (7.1) hold) implies $r_- > 0$. Notice that if $f = f_\kappa$, for some κ, then either $r_+ = 0$ or $r_- = 0$. We observe the following, which was proved earlier in ref. 26.

Proposition 7.3. For the flow to be turbulent (as defined by [K1]–[K3]), the force f must involve at least two modes with distinct wavenumbers.

We now note that the averages $\kappa_\pm > 0$ defined by

$$\frac{\sum \kappa^2 (f_\kappa, \langle u \rangle_\kappa)^\pm}{r_\pm} = \kappa_\pm^2 \tag{7.6}$$

satisfy

$$\underline{\kappa}^2 \leqslant \kappa_\pm^2 \leqslant \bar{\kappa}^2 \tag{7.7}$$

With this preliminary we can prove the following.

Proposition 7.4. Suppose $\kappa_\sigma > \bar{\kappa}$. Then

$$0 < \frac{r_-}{r_+} < 1 \tag{7.8}$$

and

$$\kappa_\sigma^2 = \kappa_-^2 + \kappa_-(\kappa_+ + \kappa_-)\,\frac{\kappa_+/\kappa_- - 1}{1 - r_-/r_+} \tag{7.9}$$

Proof. We know from the previous discussion that $r_+, r_- > 0$, and in particular, that κ_+ and κ_- are well-defined. We rewrite (7.4) as

$$\nu\langle\|u\|^2\rangle + r_- = r_+ \tag{7.10}$$

Since $f \neq 0$, we see from (2.15) that $\langle\|u\|^2\rangle > 0$. Thus $0 < r_- < r_+$, which proves (7.8).

Using (7.6) and the definition of κ_σ, we rewrite (7.2) as

$$\kappa_\sigma^2 \nu\langle\|u\|^2\rangle + \kappa_-^2 r_- = \kappa_+^2 r_+ \tag{7.11}$$

Subtracting κ_σ^2 times (7.10) from (7.11), we have

$$\kappa_\sigma^2 (r_+ - r_-) + \kappa_-^2 r_- = \kappa_+^2 r_+ \tag{7.12}$$

Solving for κ_σ^2 and rearranging, we find

$$\kappa_\sigma^2 = \frac{\kappa_+^2 r_+ - \kappa_-^2 r_-}{r_+ - r_-}$$

$$= \kappa_-^2 + \frac{(\kappa_+^2 - \kappa_-^2)\, r_+}{r_+ - r_-}$$

$$= \kappa_-^2 + (\kappa_+ + \kappa_-)\,\frac{\kappa_+ - \kappa_-}{1 - r_-/r_+}$$

from which (7.13) follows readily. ∎

Theorem 7.5. Suppose $\kappa_\sigma > \bar{\kappa}$. Then (7.1) is equivalent to

$$\kappa_+/\kappa_- - 1 \gg 1 - r_-/r_+ > 0 \qquad \text{and} \qquad r_+, r_- > 0 \tag{7.13}$$

Proof. Using (7.7) in (7.13) we find

$$\underline{\kappa}^2 + 2\underline{\kappa}^2\,\frac{\kappa_+/\kappa_- - 1}{1 - r_-/r_+} \leqslant \kappa_\sigma^2 \leqslant \bar{\kappa}^2 + 2\bar{\kappa}^2\,\frac{\kappa_+/\kappa_- - 1}{1 - r_-/r_+}$$

This gives

$$\left(\frac{\underline{\kappa}}{\bar{\kappa}}\right)^2\left[1+2\,\frac{\kappa_+/\kappa_- -1}{1-r_-/r_+}\right]\leqslant\left(\frac{\kappa_\sigma}{\bar{\kappa}}\right)^2\leqslant 1+2\,\frac{\kappa_+/\kappa_- -1}{1-r_-/r_+}$$

which provides the desired equivalence. ∎

Remark 7.6. If f involves exactly two wavenumbers κ_{lo} and κ_{hi}, then, regardless of the invariant measure used, the following conditions are by Theorem 7.5 necessary for (7.1) to hold

$$\kappa_{\mathrm{hi}}=\kappa_+,\qquad \kappa_{\mathrm{lo}}=\kappa_- \tag{7.14}$$

and

$$-r_-=(f_{\kappa_{\mathrm{lo}}},\langle u\rangle_{\kappa_{\mathrm{lo}}})<0<(f_{\kappa_{\mathrm{hi}}},\langle u\rangle_{\kappa_{\mathrm{hi}}})=r_+ \tag{7.15}$$

If, however, (7.14) and (7.15) hold, then by Theorem 7.5

$$\kappa_{\mathrm{hi}}/\kappa_{\mathrm{lo}}-1\gg 1-r_-/r_+>0 \tag{7.16}$$

is equivalent to (7.1). This is the main result in ref. 26.

Notice that (7.16) as well as (7.13) together with (7.7) imply that $1-r_-/r_+\approx 0$.

Theorem 7.7. We have

$$\left(\frac{\kappa_\eta}{\kappa_0}\right)^2\leqslant\left(\frac{1}{2\pi}\right)^{2/3}\left[\left(\frac{\kappa_\sigma}{\bar{\kappa}}\right)^2-1\right]^{-1/3}G^{2/3} \tag{7.17}$$

Moreover, if the Kraichnan theory ([K1]–[K3]) holds, then

$$\left(\frac{\kappa_\eta}{\kappa_0}\right)^2\left[\ln\frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}}\right]^{-1/7}\lesssim\left(\frac{\bar{\kappa}}{\kappa_0}\right)^{2/7}G^{4/7} \tag{7.18}$$

Proof. From (5.9) we have for $\bar{\kappa}\leqslant\kappa<\kappa_\sigma$

$$\begin{aligned}\langle|AQ_\kappa u|^2\rangle-\kappa_\sigma^2\langle\|Q_\kappa u\|^2\rangle&=\kappa_\sigma^2\langle\|P_\kappa u\|^2\rangle-\langle|AP_\kappa u|^2\rangle\\&\geqslant\left[\left(\frac{\kappa_\sigma}{\kappa}\right)^2-1\right]\langle|AP_\kappa u|^2\rangle\end{aligned}$$

In particular, for $\kappa = \bar{\kappa}$, we find that

$$\langle |AP_{\bar{\kappa}} u|^2 \rangle \leqslant \left[\left(\frac{\kappa_\sigma}{\bar{\kappa}} \right)^2 - 1 \right]^{-1} \langle |AQ_{\bar{\kappa}} u|^2 \rangle \tag{7.19}$$

Using (2.19), the Cauchy–Schwarz inequality (twice), Fubini's theorem, and (7.19), we have

$$\begin{aligned} \nu \langle |Au|^2 \rangle = (f, A\langle u \rangle) = (f, \langle AP_{\bar{\kappa}} u \rangle) &\leqslant |f| \, |\langle AP_{\bar{\kappa}} u \rangle| \leqslant |f| \, \langle |AP_{\bar{\kappa}} u|^2 \rangle^{1/2} \\ &\leqslant |f| \left[\left(\frac{\kappa_\sigma}{\bar{\kappa}} \right)^2 - 1 \right]^{-1/2} \langle |Au|^2 \rangle^{1/2} \end{aligned}$$

and consequently

$$\nu^2 \langle |Au|^2 \rangle \leqslant |f|^2 \left[\left(\frac{\kappa_\sigma}{\bar{\kappa}} \right)^2 - 1 \right]^{-1} \tag{7.20}$$

Equivalent to (7.20) we have

$$\frac{\eta}{\nu^3} = \frac{\nu L^{-2} \langle |Au|^2 \rangle}{\nu^3} \leqslant \frac{|f|^2}{\nu^4 \kappa_0^4} \kappa_0^4 / L^2 \left[\left(\frac{\kappa_\sigma}{\bar{\kappa}} \right)^2 - 1 \right]^{-1}$$

so that by definitions (4.1), (4.5), (1.23)

$$\left(\frac{\kappa_\eta}{\kappa_0} \right)^6 \leqslant G^2 \frac{1}{\kappa_0^2 L^2} \left[\left(\frac{\kappa_\sigma}{\bar{\kappa}} \right)^2 - 1 \right]^{-1}$$

The estimate in (7.17) now follows from the definition of κ_0 in (1.3).

If the Kraichnan Theory holds, then from (6.27),

$$\kappa_\sigma \sim \kappa_\eta (\ln \kappa_\eta / \underline{\kappa}_{\mathrm{i}})^{-1/2}$$

which when substituted into (7.17) leads after simple algebraic manipulations to (7.18). ▮

The ratio $(\kappa_\eta/\kappa_0)^2$ is the Landau–Lifschitz asymptotic degrees of freedom.[23]

Remark 7.8. It is shown in ref. 27 that the fractal dimension of the attractor satisfies

$$\dim_F(\mathscr{A}) \leqslant c_7 \left(\frac{\kappa_\eta}{\kappa_0} \right)^2 \left[\ln \left(\frac{\kappa_\eta}{\kappa_0} \right) + 1 \right]^{1/3} \sim c_8 G^{2/3} (\ln G + 1)^{1/3}$$

(For a simpler proof see ref. 28.) The estimate in (7.18) then offers an improvement in the upper bound on the dimension of the attractor, in the case of fully developed turbulence. Liu gave an example in ref. 29 where

$$\dim_F(\mathscr{A}) \geqslant cG^{2/3}$$

for f taken to be some special eigenvector of A. This is reconciled with (7.18) by noting that, according to 7.3, Liu's flow could not display the universal features of fully developed turbulence as discussed in this paper.

APPENDIX A

Remark A.1. Consider a fluid in three dimensions. Let $\rho(x, t)$ denote the density. Given $N \geqslant 0$ and a continuous function ψ on $\mathbb{R}^n$, where $n = 3(1+\#)$ and $\#$ is the cardinality of $\{\alpha\colon |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \leqslant N,\ \alpha_j \in \mathbb{Z}^+ \cup \{0\}\}$ we denote and define

$$\psi_{\text{pm}} = \text{quantity of } \psi \text{ per (unit of) mass}$$
$$\stackrel{\text{def}}{=} \lim_{l \to \infty} \frac{\int_{[-l,l]^3} \psi\left(x, \left(\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}\, \partial x_3^{\alpha_3}}\, u(x)\right)_{|\alpha| \leqslant N}\right) \rho(x, t)\, d^3x}{\int_{[-l,l]^3} \rho(x, t)\, d^3x}$$

provided that the integral makes sense and the limit exists. For an incompressible fluid ρ is constant, so that

$$\psi_{\text{pm}} = \lim_{l \to \infty} \frac{1}{(2l)^3} \int_{[-l,l]^3} \psi\, d^3x$$

For a two-dimensional flow

$$u = u(x_1, x_2) = (u_1, u_2, 0)$$

If, moreover, ψ is also independent of x_3, then

$$\psi_{\text{pm}} = \lim_{l \to \infty} \frac{1}{(2l)^2} \int_{[-l,l]^2} \psi\, d^2x$$

Choose n such that

$$nL \leqslant l \leqslant (n+1)\, L \tag{A.1}$$

and M such that $|\psi| \leqslant M$. If ψ and u are periodic of period L in x, we have

$$\int_{[-l,l]^2} \psi\, d^2x = \int_{[-l,l]^2 \setminus [-nL,nL]^2} \psi\, d^2x + (2n)^2 \int_{(0,L)^2} \psi\, d^2x$$

and

$$\left| \int_{[-l,l]^2 \setminus [-nL,nL]^2} \psi\, d^2x \right| \leqslant 4(2n+1)\, ML^2$$

It follows that

$$\begin{aligned}
&\left| \frac{1}{(2l)^2} \left[\int_{[-l,l]^2 \setminus [-nL,nL]^2} \psi\, d^2x + (2n)^2 \int_{(0,L)^2} \psi\, d^2x \right] - \frac{1}{L^2} \int_{[0,L]^2} \psi\, d^2x \right| \\
&\quad \leqslant \frac{4(2n+1)\, ML^2}{(2l)^2} + \frac{|(2n)^2 L^2 - (2l)^2|}{(2l)^2} \left| \frac{1}{L^2} \int_{[0,L]^2} \psi\, d^2x \right| \\
&\quad \leqslant \frac{3(nL)\, ML}{l^2} + \frac{|(nL-l)(nL+l)|}{l^2} \left| \frac{1}{L^2} \int_{[0,L]^2} \psi\, d^2x \right| \\
&\quad \leqslant \frac{3ML}{l} + \frac{L(nL+l)}{l^2} \left| \frac{1}{L^2} \int_{[0,L]^2} \psi\, d^2x \right| \\
&\quad \leqslant \frac{3ML}{l} + \frac{2L}{l} \left| \frac{1}{L^2} \int_{[0,L]^2} \psi\, d^2x \right|
\end{aligned}$$

Taking the limit as $l \to \infty$, we have

$$\psi_{\mathrm{pm}} = \frac{1}{L^2} \int_{[0,L]^2} \psi\, d^2x$$

Notation

L	length of square spatial domain Ω
A	$-\Delta$, with periodic boundary conditions on Ω
λ_j, $j = 0, 1, 2,...$	eigenvalues of A in increasing order, counted with multiplicity
w_j, $j = 0, 1, 2,...$	eigenfunctions of A corresponding to λ_j
H	$\{u \in L^2(\Omega) : u = \mathbb{R}^2$-valued trigonometric polynomial, $\nabla \cdot u = 0$, and $\int_\Omega u(x)\, dx = 0\}$
V	$\mathcal{D}_{A^{1/2}}$

P_κ	orthogonal projector: $H \to \operatorname{span}\{w_j \mid \lambda_j \leqslant \kappa^2\}$
Q_κ	$I - P_\kappa$
$u_{\kappa,\kappa'}$	$(P_{\kappa'} - P_\kappa)\, u$, for $u \in H$
κ_0	$\lambda_0^{1/2} = 2\pi/L$
p_κ	$u_{\kappa_0,\kappa}$
q_κ	$u_{\kappa,\infty}$
$\mathscr{A}$	the global attractor for the Navier–Stokes equations (NSE)
f	body force in abstract form of NSE (1.1)
ν	kinematic viscosity
$\lvert\cdot\rvert$	norm in $L^2(\Omega)$
$\lVert u\rVert$	$\lvert A^{1/2}u\rvert$
G	generalized Grashof number, $\frac{\lvert f\rvert}{\nu^2\kappa_0^2}$
G_*	associated Grashof number, $\frac{\lvert A^{-1/2}f\rvert}{\nu^2\kappa_0}$
$\mathfrak{E}_\kappa^{\rightarrow}(u)$	$\frac{-1}{L^2}\,(B(p_\kappa, p_\kappa), Aq_\kappa)$, rate of enstrophy transfer from low to high wavenumbers at κ
$\mathfrak{E}_\kappa^{\leftarrow}(u)$	$\frac{-1}{L^2}\,(B(q_\kappa, q_\kappa), Ap_\kappa)$, rate of enstrophy transfer from high to low wavenumbers at κ
$\mathfrak{E}_\kappa$	$\mathfrak{E}_\kappa^{\rightarrow} - \mathfrak{E}_\kappa^{\leftarrow}$, net rate of enstrophy transfer at wavenumber κ
$\mathfrak{e}_\kappa^{\rightarrow}(u)$	$-(B(p_\kappa, p_\kappa), q_\kappa)$, rate of energy transfer from low to high wavenumbers at κ
$\mathfrak{e}_\kappa^{\leftarrow}(u)$	$-(B(q_\kappa, q_\kappa), p_\kappa)$, rate of energy transfer from high to low wavenumbers at κ
$\mathfrak{e}_\kappa$	$\mathfrak{e}_\kappa^{\rightarrow} - \mathfrak{e}_\kappa^{\leftarrow}$, net rate of energy transfer at wavenumber κ
inertial range	$[\underline{\kappa}_{\rm i}, \bar{\kappa}_{\rm i}]$
dissipation range	$[\kappa_{\rm d}, \infty)$
κ_σ	$\left(\frac{\langle\lvert Au\rvert^2\rangle}{\langle\lVert u\rVert^2\rangle}\right)^{1/2}$
κ_τ	$\left(\frac{\langle\lVert u\rVert^2\rangle}{\langle\lvert u\rvert^2\rangle}\right)^{1/2}$
η	$\frac{\nu}{L^2}\,\langle\lvert Au\rvert^2\rangle = \frac{1}{L^2}\,\langle(f, Au)\rangle = \frac{1}{L^2}\,(f, A\langle u\rangle)$
κ_η	$\left(\frac{\eta}{\nu^3}\right)^{1/6}$
ϵ	$\frac{\nu}{L^2}\,\langle\lVert u\rVert^2\rangle = \frac{1}{L^2}\,(f, \langle u\rangle)$
$\bar{\kappa}$	smallest κ such that $P_\kappa f = f$
$\underline{\kappa}$	largest κ such that $Q_\kappa f = f$

$\bar{\kappa}_{\mathrm{d}}$	$\frac{\kappa_\eta}{\pi(2c_{\mathrm{Kr}})^{1/2}}$
$\underline{\kappa}_\sigma$	$\left(\frac{\langle\|AP_{\underline{\kappa}}u\|^2\rangle}{\langle\|P_{\underline{\kappa}}u\|^2\rangle}\right)^{1/2}$
$\bar{\kappa}_\sigma$	$\left(\frac{\langle\|AQ_{\bar{\kappa}}u\|^2\rangle}{\langle\|Q_{\bar{\kappa}}u\|^2\rangle}\right)^{1/2}$
c_{Kr}	Kraichnan constant; it is assumed that $c_{\mathrm{Kr}} \sim 1$
$a \ll b$	$\frac{a}{b} < \delta$ for some small $\delta \in (0, 1)$, a/b should be nondimensional. The value of δ shall remain unspecified, and may vary from one statement involving $\ll$ to the next. In some instances it may depend on the value of δ chosen earlier, but is always independent of physical parameters such as ν, f and L.
$a \approx b$	$\|a/b - 1\| \ll 1$, a/b should be nondimensional
$a \lesssim b$	$\exists$ a universal constant c so that $a/b \leqslant c$, a/b should be nondimensional
$a \sim b$	$a \lesssim b$ and $b \lesssim a$

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